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Unit 2 : Properties of Functions - Lesson 2

The Second Derivative Test, Concavity and Points of Inflexion

LI

- Determine maxima/minima using the Second Derivative Test.
- Find Points of Inflexion.

SC

- Differentiation.

The Second Derivative Test

If $f'(p) = 0$, i.e. if $(p, f(p))$ is a stationary point of a function f :

- if $f''(p) > 0$, then $(p, f(p))$ is a local minimum turning point.
- if $f''(p) < 0$, then $(p, f(p))$ is a local maximum turning point.
- if $f''(p) = 0$, then further analysis is needed (nature table).

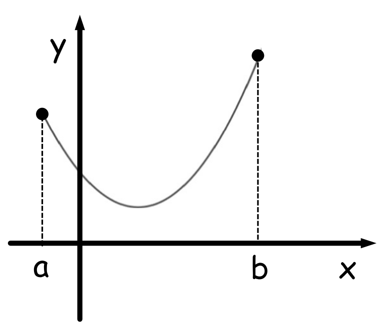
Concavity

Let I be an interval of real numbers.

- If $f''(x) \geq 0 \ \forall x \in I$, f is concave up on I (and vice versa).
- If $f''(x) \leq 0 \ \forall x \in I$, f is concave down on I (and vice versa).

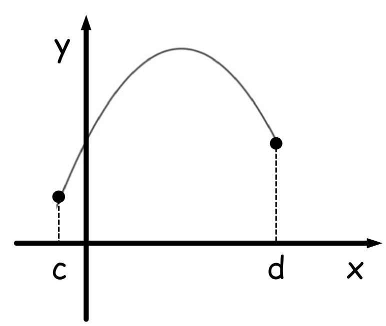
- If f is concave up on an interval, the tangent line at any point on this interval lies below the graph of f in this interval.
- If f is concave down on an interval, the tangent line at any point on this interval lies above the graph of f in this interval.

$y = f(x)$



f is concave up on $[a, b]$

$y = g(x)$



g is concave down on $[c, d]$

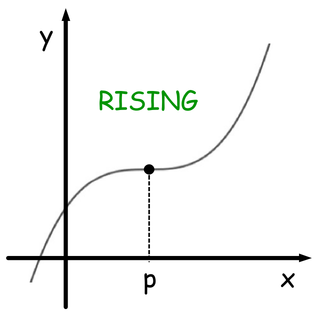
Points of Inflexion

A point of inflexion (P of I) is a point where the concavity of a function changes from concave up to down (or vice versa)

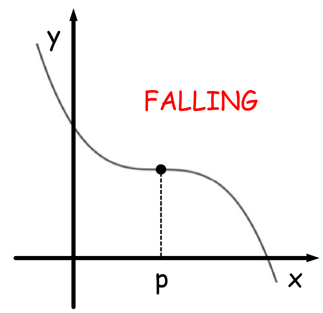
If $x = p$ is a point of inflexion, then $f''(p) = 0$ or $f''(p)$ does not exist

Types of Points of Inflexion

Horizontal :

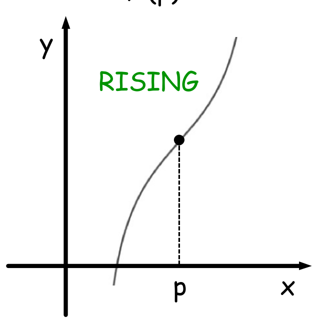


$f'(p) = 0$

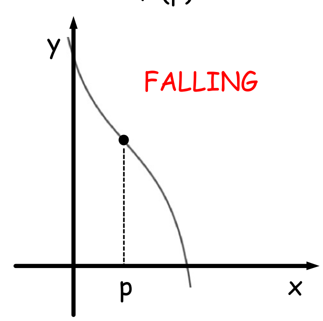


Non-horizontal :

$f'(p) > 0$



$f'(p) < 0$



WARNING : If $f''(p) = 0$, there may be a max., min. or P of I at $x = p$

When looking for P of I, solve $f''(x) = 0$; if $x = p$ is a solution of this equation, check the sign of f'' either side of p

Example 1

Find the stationary points of $f(x) = (1/3)x^3 - 2x^2 - 12x$ and determine their nature.

$$f(x) = (1/3)x^3 - 2x^2 - 12x$$

$$\therefore \quad f'(x) = x^2 - 4x - 12$$

$$\therefore \quad \underline{f''(x) = 2x - 4}$$

For SPs, $f'(x) = 0$. So,

$$x^2 - 4x - 12 = 0$$

$$\Rightarrow \quad (x - 6)(x + 2) = 0$$

$$\Rightarrow \quad \underline{x = 6, x = -2}$$

$x = 6 \Rightarrow y = -72;\ f''(6) = 8 > 0$

$\boxed{(6, -72) \text{ is a local minimum TP}}$

$x = -2 \Rightarrow y = 40/3;\ f''(-2) = -8 < 0$

$\boxed{(-2, 40/3) \text{ is a local maximum TP}}$

Example 2

Identify the points of inflexion for $f(x) = \cos x \ (0 < x < 2\pi)$ and state whether the gradient of the tangent at each of the inflexion points is positive, negative or zero.

$$f(x) = \cos x$$

$$\therefore \quad f'(x) = -\sin x$$

$$\therefore \quad \underline{f''(x) = -\cos x}$$

For points of inflexion, $f''(x) = 0$. So,

$$-\cos x = 0$$

$$\Rightarrow \quad \underline{x = \pi/2, x = 3\pi/2}$$

$x = \pi/2 \Rightarrow y = 0;\ f'(\pi/2) = -1 < 0$

$\boxed{\text{P of I } (\pi/2, 0) \text{ has a negative gradient at } x = \pi/2}$

$x = 3\pi/2 \Rightarrow y = 0;\ f'(3\pi/2) = 1 > 0$

$\boxed{\text{P of I } (3\pi/2, 0) \text{ has a positive gradient at } x = 3\pi/2}$

Example 3

Discuss the concavity of $f(x) = \tan x \ (-\pi/2 < x < \pi/2)$.

$$f(x) = \tan x$$

$$\therefore \quad f'(x) = \sec^2 x$$

$$\therefore \quad f''(x) = 2(\sec x) . \sec x \tan x$$

$$\Rightarrow \quad f''(x) = 2\sec^2 x \tan x$$

$$\Rightarrow \quad \underline{f''(x) = \frac{2\sin x}{\cos^3 x}}$$

If $-\pi/2 < x < 0$, $\sin x < 0$ and $\cos x > 0$; hence, $f''(x) < 0 \Rightarrow f''(x) \leq 0$.

$$\boxed{f \text{ is concave down for } -\pi/2 < x < 0}$$

If $0 < x < \pi/2$, $\sin x > 0$ and $\cos x > 0$; hence, $f''(x) > 0 \Rightarrow f''(x) \geq 0$.

$$\boxed{f \text{ is concave up for } 0 < x < \pi/2}$$

For possible P of I, $f''(x) = 0$. So,

$$\sin x = 0$$

If $-\pi/2 < x < \pi/2$, the only solution to this equation is,

$$x = 0$$

$$x = 0 \Rightarrow y = 0;\ f'(0) = 1 > 0$$

$$\boxed{(0, 0) \text{ is a non-horizontal (rising) P of I}}$$

Oct 2-23:10

AH Maths - MiA (2nd Edn.)

- pg. 70 Ex. 5.5 Q 1 a, d, e.
- pg. 73 Ex. 5.7 Q 1, 2 b, c, d,
  5 a - d.

Ex. 5.5

1 Identify the stationary points of these functions and determine their natures.

a $f(x) = 2x^2 - 12x + 3$

d $f(x) = 2x^3 + 3x^2 - 12x + 1$

e $f(x) = 3 - 24x + 18x^2 - 4x^3$

Ex. 5.7

1 Use the second derivative to prove the graph of

a $f(x) = x^2$ is always concave up

b $f(x) = \ln x$ is always concave down.

2 Discuss the concavity of the graph of

b $f(x) = x + \frac{1}{x}$ in the interval $0 < x \leq 5$

c $f(x) = \tan x$ in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$

d $f(x) = x^3 + 6$

5 Identify the points of inflexion of these curves.
In each case say whether the gradient of the tangent at that point is positive, negative or zero.

a $f(x) = x^3$

b $f(x) = x^4 - 6x^2$

c $f(x) = x^5 - 30x^3$

d $f(x) = \cot x$ in the interval $0 < x < \pi$

Answers to AH Maths (MiA), pg. 70, Ex. 5.5

1 a Min $(3, -15)$

d Min$(1, -6)$, Max $(-2, 21)$

e Min $(1, -7)$, Max $(2, -5)$

Answers to AH Maths (MiA), pg. 73, Ex. 5.7

1 a $f''(x) = 2 > 0$

b $f''(x) = -\frac{1}{x^2} < 0$

2 b Up

c Down when $x < 0$; up when $x > 0$; inflexion at $(0, 0)$

d Down when $x < 0$; up when $x > 0$; inflexion at $(0, 6)$

5 a $(0, 0)$ zero

b $(1, -5)$ negative; $(-1, -5)$ positive

c $(-3, 567)$ negative; $(0, 0)$ zero; $(3, -567)$ negative

d $\left(\frac{\pi}{2}, 0\right)$ negative