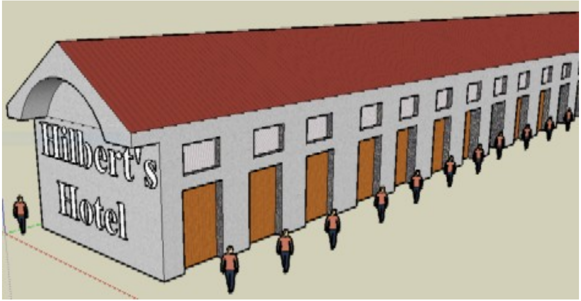
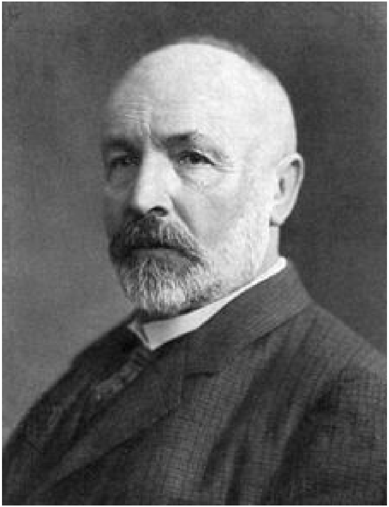
Infinity II: The Hilbert Hotel



Today we'll start by discussing how Cantor developed the notion of different sizes of infinity and, in doing so, clarified Galileo's paradox.

Later, we'll investigate some puzzles (or paradoxes) created by Hilbert.

Georg Cantor

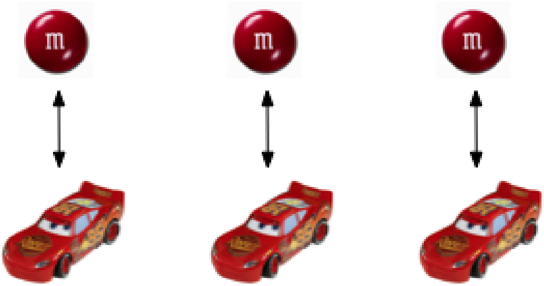


Cantor, a German mathematician, whose career was mostly in the later part of the 19th century, did important work in set theory. He formalized the idea of the size of a set, and defined what it means for one set to be larger, smaller, or the same size as another set. He used the idea Galileo discussed.

His work, applied to infinite sets, was harshly criticized by many, including some of the most famous mathematicians of the time. One, David Hilbert, strongly supported his work. We will talk about Hilbert and his puzzles based on Cantor's work later today.

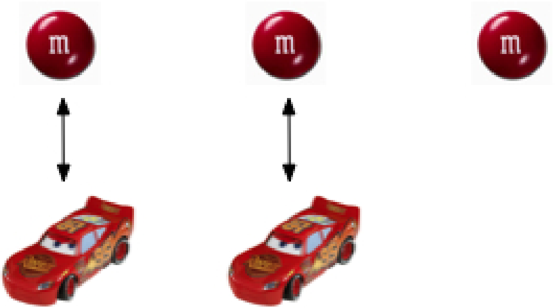
How to Compare Sizes of Sets

Kids can compare two sets before knowing their numbers by pairing off elements. For example, a very young child can understand that there are just as many M&Ms as cars in the following picture.



Roughly, two sets have the same size if one can pair off elements of one set with elements of the other, leaving no elements left out of the pairing.

One set is larger than another if the second is the same size as a subset of the first, but not vice-versa.



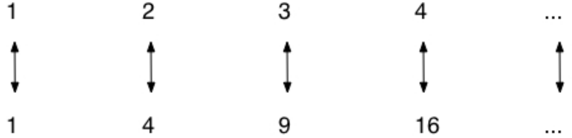
Infinite Sets

Cantor extended this idea to arbitrary sets by saying two sets have the same size, whether or not they are finite or infinite, if elements of one can be paired with elements of the other, leaving no elements out.

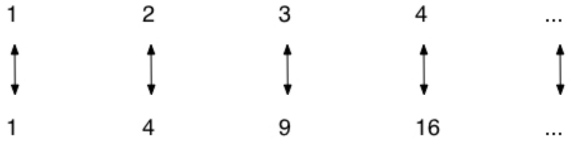
With his definition, it is true that the set of whole numbers $\{1, 2, 3, 4, 5, \ldots\}$ is the same size as the set of squares $\{1, 4, 9, 16, 25, \ldots\}$, by what Salviati says in Galileo's dialogue.

Cantor also defined a set to be infinite if it is the same size as a proper subset. It is not at all obvious that this corresponds to our intuition.

For example, the set of whole numbers is infinite, because it is the same size as the set squares.



Galileo's paradox arises from the thought that one should be able to say two sets are the same size if you can pair up elements, which makes perfect sense for finite sets. Galileo thought that the set of squares shouldn't be as large as the set of whole numbers, even though they can be paired off.



With Cantor's definition of infinity, any infinite set would be just as mysterious to Galileo.

Clicker Question

Do you think all infinite sets are the same size?

A Yes

B No

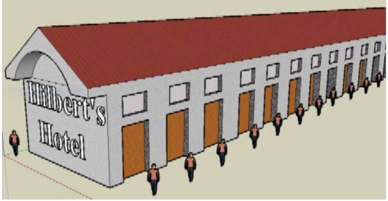
We'll discuss this next time.

David Hilbert

David Hilbert was a German mathematician who lived from 1862-1943. He was considered one of the top mathematicians in the world at the beginning of the 20th century, and he influenced many areas of mathematics. While many mathematicians derided Cantor for his work on set theory, Hilbert was a strong supporter of Cantor's work.

The puzzles we will present are due to Hilbert, hence the name of this lecture.

The Hilbert Hotel



There are several puzzles that go under the title “The Hibert Hotel.” We will discuss several of these.

The Hilbert Hotel is an unusual hotel. There is one room for every whole number. That is, there is Room 1, Room 2, and so on. It has infinitely many rooms. It would be a drag having to clean it!

Hilbert Hotel Question 1

It is a dark and stormy Friday the 13th. The Hilbert Hotel is fully booked. Late that night a traveller comes to the hotel hoping for a room. There is a sign outside the hotel saying Fully Booked. Confused that the sign does not say no vacancy, the traveller entered the lobby and inquires about a room. The hotel clerk says that in spite of every room being occupied, he can accommodate the traveller without having to kick anybody out of the hotel or have guests share rooms.

Can he do this?

Clicker Question

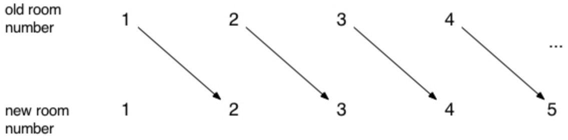
Can the hotel clerk accommodate everybody?

A Yes

B No

Answer

The hotel clerk tells the guest in room 1 to move to room 2, the guest in room 2 to move to room 3, and so on. He then tells the traveller to take room 1. The traveller gets a room, and every current guest gets a room. In general, the guest who was in Room n moves to Room $n+1$. Therefore, nobody is kicked out and nobody has to share a room.



Question 2

Two couples come to the Hilbert Hotel, which is fully booked. They hope to book 2 rooms. The clerk says he can accommodate their request. How can he do this without kicking anybody out of the hotel?

Clicker Question

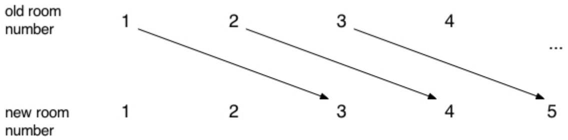
Can the hotel clerk accommodate everybody, when two couples come to a fully booked Hilbert Hotel?

A Yes

B No

Answer

The clerk asks the guest in Room 1 to move to Room 3, the guest in Room 2 to move to Room 4, and so on. This frees up Rooms 1 and 2, and he puts the two couples in those two rooms. Again, the travelers are accommodated without kicking anybody out of the hotel and without guests having to share rooms. In general, the person who was in Room n gets moved to Room $n+2$.



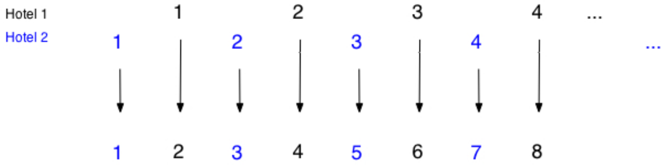
Question 3

There are two fully booked Hilbert Hotels in town. During a lightning storm one of the hotels burns down. The manager calls the other hotel's manager and asks if he can accommodate the first hotel's guests. The second manager says sure, I can do that.

How can the second manager accommodate all the guests from the first hotel without kicking any of the current guests out of the hotel?

Answer

This is more difficult, but the manager figures out how to handle the request. He asks the guest in Room 1 to move to Room 2, the guest in Room 2 to move to Room 4, and so on. In general, the guest in Room n moves to Room $2n$. This frees up Rooms 1, 3, 5, 7, and so on. He then puts the first new guest in Room 1, the second in Room 3, and so on. In general, the guest who was in Room n of the first Hilbert Hotel occupies Room $2n-1$ in the second hotel. Everybody gets a room and nobody has to share.



In fact, if a town has any (finite) number of fully booked Hilbert Hotels, all of the guests could be accommodated in just one of them without kicking anybody out or making anybody share.

If you are curious about this, think first about the case of three hotels, and see if you can see a pattern for how to do this.

Question ∞

A large town has one Hilbert Hotel for every whole number; that is, there is a Hilbert Hotel 1, a Hilbert Hotel 2, and so on. In a freak storm all the hotels burn down. Luckily a brand new Hilbert Hotel was been built in the next town, and is now open and ready to take on customers. Can all the guests at the destroyed hotels be accommodated in the new Hilbert Hotel?

Clicker Question

What do you think? Can one Hilbert Hotel accommodate all the guests of infinitely many Hilbert Hotels?

A Yes

B No

Answer

This is not easy, but can be done! In fact, here is a way to accommodate everybody and have tons of rooms left! He assigns the person who was in Room n of hotel m Room number $2^n \cdot 3^m$. For example, the guest in Room 1 of hotel 1 gets Room $2 \cdot 3 = 6$. The guest in Room 3 of hotel 2 gets Room

$$2^3 \cdot 3^2 = (2 \cdot 2 \cdot 2) \cdot (3 \cdot 3) = 8 \cdot 9 = 72$$

This allows everybody to have a room with no two guests being assigned to the same room.

This explanation uses a fact about whole numbers and the uniqueness of factorization into prime numbers. The numbers 2 and 3 are primes. No number can be factored into a product of primes (e.g., 2 and 3) in two different ways. For example, we can't have a product of five 2s and seven 3s equal to a product of six 2s and six 3s. This is why nobody has to share a room. This scheme leaves many rooms (infinitely many!) vacant. For example, Rooms 1, 5, 7, 10, 11, 13, 14 are but a few of those left vacant.

Here is a bit of a diagram to show what is happening for some of the hotels and rooms.

Room # \ Hotel #	1	2	3	4
1	6	18	54	162
2	12	36	108	324
3	24	72	216	648
4	48	144	432	1296

Next Time

We will get back to the paradoxes discussed on Monday and we will also address the question, posed earlier, whether all infinite sets have the same size, according to Cantor's notion of size.

We will also see some other instances of infinity and some videos about them.