20 / 12 / 17

Unit 2 : Applications of Calculus - Lesson 1

Rectilinear Motion

LI

- Solve problems involving motion in a straight line.

SC

- Differentiation.
- Integration.

Rectilinear means in a straight line

The displacement (denoted variously by x, s, or r) of a particle is its position from the origin of a coordinate system

The velocity (denoted by v) of a particle is the time-derivative of the particle's displacement :

$$v = \frac{dx}{dt} = \dot{x}$$

The acceleration (denoted by a) of a particle is the time-derivative of the particle's velocity :

$$a = \frac{dv}{dt} = \dot{v}$$

Unless otherwise stated, the following units are assumed :

- Displacement - metres (m).
- Velocity - metres per second ($m\ s^{-1}$).
- Acceleration - metres per second squared ($m\ s^{-2}$).

Connections Between Displacement, Velocity and Acceleration

$$\begin{tikzcd} x(t) \arrow[r, shift left, "\frac{d}{dt}"] & v(t) \arrow[l, shift left, "\int"] \arrow[r, shift left, "\frac{d}{dt}"] & a(t) \arrow[l, shift left, "\int"] \end{tikzcd}$$

Useful Terminology

Constant - unchanging

At rest - velocity $= 0$

Initially - time starts at 0

At origin - displacement $= 0$

Unless otherwise stated, a particle will be assumed to commence its motion from the origin at $t = 0$

Example 1

A particle has velocity $v(t) = 3t^2 - 4t$. Initially, the particle has a displacement of -3 m from the origin of the coordinate system.

Calculate the particle's :

(a) displacement at time 4 s.
(b) displacement from the origin at $t = 1$ s.
(c) acceleration at $t = 2/3$ s.

(a) $$r(t) = \int v(t)\ dt$$

$$\therefore \quad r(t) = \int (3t^2 - 4t)\ dt$$

$$\Rightarrow \quad r(t) = t^3 - 2t^2 + C$$

$r(0) = -3$ gives,

$$-3 = 0^3 - 2(0)^2 + C$$

$$\Rightarrow \quad \underline{C = -3}$$

$$\therefore \quad \underline{r(t) = t^3 - 2t^2 - 3}$$

$$\therefore \quad r(4) = 4^3 - 2(4)^2 - 3$$

$$\Rightarrow \quad \boxed{r(4) = 29\ \text{m}}$$

(b) $$r(1) = 1^3 - 2(1)^2 - 3$$

$$\Rightarrow \quad \boxed{r(1) = -4\ \text{m}}$$

(c) $$a(t) = \frac{d}{dt}\, v(t)$$

$$\Rightarrow \quad a(t) = \frac{d}{dt}(3t^2 - 4t)$$

$$\Rightarrow \quad \underline{a(t) = 6t - 4}$$

$$\therefore \quad a(2/3) = 6(2/3) - 4$$

$$\Rightarrow \quad \boxed{a(2/3) = 0\ \text{m s}^{-2}}$$

Jun 8-12:56

Example 2

A particle has acceleration $a(t) = \sin 2t$. Initially, the particle is at rest at the origin of the coordinate system.

Calculate the :

(a) particle's velocity at time $\pi/2$ s.
(b) particle's displacement from the origin at $t = \pi/2$ s.
(c) times at which the acceleration is zero.

(a) $v(t) = \int a(t)\, dt$

$\therefore \quad v(t) = \int \sin 2t \, dt$

$\Rightarrow \quad v(t) = -1/2 \cos 2t + C$

$v(0) = 0$ gives,

$0 = -1/2\,(1) + C$

$\Rightarrow \quad \underline{C = 1/2}$

$\therefore \quad \underline{v(t) = 1/2 - 1/2 \cos 2t}$

$\therefore \quad v(\pi/2) = 1/2 - 1/2 \cos \pi$

$\Rightarrow \quad \boxed{v(\pi/2) = 1 \text{ m s}^{-1}}$

(b) $r(t) = \int v(t)\, dt$

$\therefore \quad r(t) = \int (1/2 - 1/2 \cos 2t)\, dt$

$\Rightarrow \quad r(t) = t/2 - 1/4 \sin 2t + D$

$r(0) = 0$ gives,

$0 = 0/2 - 1/4\,(0) + D$

$\Rightarrow \quad \underline{D = 0}$

$\therefore \quad \underline{r(t) = t/2 - 1/4 \sin 2t}$

$\therefore \quad r(\pi/2) = \pi/4 - 1/4 \sin \pi$

$\Rightarrow \quad \boxed{r(\pi/2) = \pi/4 \text{ m}}$

(c) $a(t) = 0$

$\therefore \quad \sin 2t = 0$

$\Rightarrow \quad \boxed{t = (n\pi/2) \text{ s} \quad (n = 0, 1, 2, 3, \ldots)}$

Jun 8-12:56

Example 3

A particle has displacement governed by the equation
$s(t) = t^3 - 9t^2 + 15t - 2.$

When is the velocity decreasing ?

$$v(t) = \frac{d}{dt}\, s(t)$$

$$\Rightarrow \quad v(t) = \frac{d}{dt}\,(t^3 - 9t^2 + 15t - 2)$$

$$\Rightarrow \quad \underline{v(t) = 3t^2 - 18t + 15}$$

The velocity is decreasing when $\frac{d}{dt}\, v(t)$ is negative, i.e.

when the acceleration is negative.

$$a(t) = \frac{d}{dt}\, v(t)$$

$$\Rightarrow \quad a(t) = \frac{d}{dt}\,(3t^2 - 18t + 15)$$

$$\Rightarrow \quad \underline{a(t) = 6t - 18}$$

$$a(t) < 0$$

$$\Rightarrow \quad 6t - 18 < 0$$

$$\Rightarrow \quad 6t < 18$$

$$\Rightarrow \quad \boxed{t < 3\ s}$$

Jun 8-12:56

AH Maths - MiA (2^{nd} Edn.)

- pg. 187-9 Ex. 11.1 Q 1 a - f, 2 a, 3 a, 4, 5 a, 11, 14.

Ex. 11.1

1 Given that particles are moving in a straight line according to the equations given below, calculate for each the velocity and acceleration

i after t seconds ii after 3 s.

a $x = 3t^2 + t + 1$ b $x = \dfrac{2t}{t+1}$ c $x = \sqrt{5t+1}$

d $x = t + e^{3-t}$ e $x = 2\sqrt{2}\sin\dfrac{\pi}{12}t$ f $x = t + \dfrac{1}{t}$

2 The movement of particles can be modelled by the equations below.
Find the displacement and the acceleration the moment each particle comes to rest for the first time ($v = 0$).

a $x = 180t - 3t^2$

3 A particle is moving in a straight line. Its velocity can be calculated from $v(t) = 3t^2 + 1$ where t is the time in seconds. Initially the particle is 3 m from the origin.

a Find an expression for $x(t)$, the displacement of the particle at time t.

4 A particle is moving so that the acceleration in ms^{-2} is modelled by $a = 3\sin 2t$ where t is the time in seconds since observations began.
Initially the particle is at rest and 1 m from the origin.
[That is, at $t = 0$, $v = 0$ and $s = 1$.]

a Find an expression for

i the velocity at time t ii the displacement at time t.

b Calculate the displacement, velocity and acceleration of the particle when t is $\frac{\pi}{4}$ s.

5 The equation of motion of a particle is $x = t^2 - \sin t$.

a Show that the particle is always accelerating.

11 The height, s metres, reached in t seconds by a body thrown vertically upwards with initial velocity v_0 ms^{-1} is given by the formula $s = v_0 t - \frac{1}{2}gt^2$ where g is the acceleration due to gravity.
Find an expression for

a v_t, the velocity at time t

b the times when $s = 0$ m

c the time when the body reaches its maximum height

d the maximum height.

14 A particle, starting from rest, proceeds in a straight line. Its acceleration after t seconds is given by $a = 4\sec^2 t$ unit s^{-2} where $0 \le t \le 1$. Calculate the velocity after 0.5 s and the distance travelled after $\frac{\pi}{4}$ s.

Answers to AH Maths (MiA), pg. 187-9, Ex. 11.1

1 a i $\dot{x} = 6t + 1; \ddot{x} = 6$ ii $19\text{ ms}^{-1}; 6\text{ ms}^{-2}$

b i $\dot{x} = \dfrac{2}{(t+1)^2}; \ddot{x} = \dfrac{-4}{(t+1)^3}$ ii $\frac{1}{8}\text{ ms}^{-1}; -\frac{1}{16}\text{ ms}^{-2}$

c i $\dot{x} = \dfrac{5}{2(5t+1)^{\frac{1}{2}}}; \ddot{x} = \dfrac{-25}{4(5t+1)^{\frac{3}{2}}}$

ii $\frac{5}{8}\text{ ms}^{-1}; -\frac{25}{256}\text{ ms}^{-2}$

d i $\dot{x} = 1 - e^{3-t}; \ddot{x} = e^{3-t}$ ii $0\text{ ms}^{-1}; 1\text{ ms}^{-2}$

e i $\dot{x} = \dfrac{\sqrt{2}\pi}{6}\cos\left(\dfrac{\pi t}{12}\right); \ddot{x} = -\dfrac{\sqrt{2}\pi^2}{72}\sin\left(\dfrac{\pi t}{12}\right)$

ii $\frac{\pi}{6}\text{ ms}^{-1}; -\frac{\pi^2}{72}\text{ ms}^{-2}$

f i $\dot{x} = 1 - t^{-2}; \ddot{x} = 2t^{-3}$ ii $\frac{8}{9}\text{ ms}^{-1}; \frac{2}{27}\text{ ms}^{-2}$

2 a $2700\text{ m}; -6\text{ ms}^{-2}$

3 a $x(t) = t^3 + t + 3$

4 a i $v = -\frac{3}{2}\cos 2t + \frac{3}{2}$ ii $x = -\frac{3}{4}\sin 2t + \frac{3}{2}t + 1$

b $x = \left(\frac{1}{4} + \frac{3\pi}{8}\right)\text{ m}; v = \frac{3}{2}\text{ ms}^{-1}; a = 3\text{ ms}^{-2}$

5 a $x = t^2 - \sin t \Rightarrow v = 2t - \cos t \Rightarrow a = 2 + \sin t;$

since $-1 \leqslant \sin t \leqslant 1 \Rightarrow 1 \leqslant 2 + \sin t \leqslant 3$

That is, acceleration always positive.

11 a $v = v_0 - gt$ **b** $t = 0; t = \dfrac{2v_0}{g}$

c $t_{\text{max}} = \dfrac{v_0}{g}$ **d** $s_{\text{max}} = \dfrac{{v_0}^2}{2g}$

14 a 2·19 m/s **b** 1·39 m

Jun 1-11:51