

# AH Mathematics

## Methods in Algebra and Calculus

### Practice Assessment

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## Solutions

## Methods in Algebra and Calculus Assessment Standard 1.1

$$1 \quad \frac{x^2 + 24}{x^3 + 8x} = \frac{x^2 + 24}{x(x^2 + 8)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 8}; \text{ hence,}$$

$$x^2 + 24 = A(x^2 + 8) + (Bx + C)x$$

$$x = 0 \Rightarrow 24 = 8A \Rightarrow A = 3$$

$$x^2 \text{ coefficients : } 1 = A + B \Rightarrow B = -2$$

$$x \text{ coefficients : } 0 = C$$

Thus,

$$\frac{x^2 + 24}{x^3 + 8x} = \frac{3}{x} - \frac{2x}{x^2 + 8}$$

## Methods in Algebra and Calculus Assessment Standard 1.2

$$2 \quad f(x) = e^{x^2 - 6x}$$

$$\Rightarrow f'(x) = e^{x^2 - 6x} \cdot \frac{d}{dx} e^{x^2 - 6x}$$

$$\Rightarrow f'(x) = (2x - 6)e^{x^2 - 6x} \text{ or } 2(x - 3)e^{x^2 - 6x}$$

$$3 \quad y = \sqrt{\tan 5x} = (\tan 5x)^{\frac{1}{2}}; \text{ hence,}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} (\tan 5x)^{-\frac{1}{2}} \cdot \frac{d}{dx} \tan 5x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} (\tan 5x)^{-\frac{1}{2}} \cdot 5 \sec^2 5x$$

$$\Rightarrow \frac{dy}{dx} = \frac{5}{2} \sec^2 5x \cdot (\tan 5x)^{-\frac{1}{2}} \text{ or } \frac{5 \sec^2 5x}{2 \sqrt{\tan 5x}}$$

$$4 \quad \text{a) } f(x) = 4x^5 \sin x$$

$$\therefore f'(x) = 4 \cdot 5x^4 \cdot \sin x + 4 \cdot x^5 \cdot \cos x$$

$$\Rightarrow f'(x) = 20x^4 \sin x + 4x^5 \cos x \text{ or } 4x^4(5 \sin x + x \cos x)$$

$$\text{b)} \quad g(x) = \frac{3x + 2}{x - 4}$$

$$\therefore g'(x) = \frac{3(x - 4) - 1(3x + 2)}{(x - 4)^2}$$

$$\Rightarrow g'(x) = \frac{3x - 12 - 3x - 2}{(x - 4)^2}$$

$$\Rightarrow g'(x) = -\frac{14}{(x - 4)^2}$$

$$5 \quad f(x) = \sin^{-1}(6x)$$

$$\therefore f'(x) = \frac{1}{\sqrt{1 - (6x)^2}} \cdot \frac{d}{dx} 6x$$

$$\Rightarrow f'(x) = \frac{6}{\sqrt{1 - 36x^2}}$$

$$6 \quad x^5 y + xy^2 = 11$$

$$\therefore 5x^4 \cdot y + x^5 \cdot y' + 1 \cdot y^2 + x \cdot 2y y' = 0$$

$$\Rightarrow 5x^4 y + y^2 + x^5 y' + 2xy y' = 0$$

$$\Rightarrow y'(x^5 + 2xy) = -(5x^4 y + y^2)$$

$$\Rightarrow y' = -\frac{5x^4 y + y^2}{x^5 + 2xy} \quad \text{or} \quad -\frac{y(5x^4 + y)}{x(x^4 + 2y)}$$

$$7 \quad x = 5t \Rightarrow \dot{x} = 5; \quad y = 12t - 2t^2 \Rightarrow \dot{y} = 12 - 4t. \quad \text{When } t = 2,$$

$$\dot{x} = 5 \quad \text{and} \quad \dot{y} = 4.$$

$$\text{Speed} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\therefore \text{Speed} = \sqrt{5^2 + 4^2}$$

$$\Rightarrow \text{Speed} = \sqrt{41}$$

### Methods in Algebra and Calculus Assessment Standard 1.3

8 a) Let  $I = \int \frac{8}{\sqrt{1 - (3x)^2}} dx$ . The substitution  $u = 3x$  gives,

$$u = 3x \Rightarrow \frac{du}{dx} = 3 \Rightarrow dx = \frac{1}{3} du$$

$$\therefore I = \frac{8}{3} \int \frac{1}{\sqrt{1 - u^2}} du$$

$$\Rightarrow I = \frac{8}{3} \sin^{-1} u + C$$

$$\Rightarrow I = \frac{8}{3} \sin^{-1}(3x) + C$$

b) Let  $I = \int \frac{7}{14x - 1} dx$ . The substitution  $u = 14x - 1$  gives,

$$u = 14x - 1 \Rightarrow \frac{du}{dx} = 14 \Rightarrow dx = \frac{1}{14} du$$

$$\therefore I = \frac{1}{2} \int \frac{1}{u} du$$

$$\Rightarrow I = \frac{1}{2} \ln|u| + C$$

$$\Rightarrow I = \frac{1}{2} \ln|14x - 1| + C$$

c) Let  $I = \int_0^{\frac{\pi}{20}} \sec^2 5x dx$ . The substitution  $u = 5x$  gives,

$$u = 5x \Rightarrow \frac{du}{dx} = 5 \Rightarrow dx = \frac{1}{5} du$$

$$x = 0 \Rightarrow u = 0; x = \frac{\pi}{20} \Rightarrow u = \frac{\pi}{4}$$

$$\therefore I = \frac{1}{5} \int_0^{\frac{\pi}{4}} \sec^2 u \ du$$

$$\Rightarrow I = \frac{1}{5} [\tan u]_0^{\frac{\pi}{4}}$$

$$\Rightarrow I = \frac{1}{5} \left( \tan \frac{\pi}{4} - \tan 0 \right)$$

$$\Rightarrow I = \frac{1}{5}$$

9      Let  $I = \int \frac{\sin x}{\cos^7 x} dx.$

$$u = \cos x \Rightarrow \frac{du}{dx} = \sin x \Rightarrow du = \sin x \ dx$$

$$\therefore I = \int \frac{1}{u^7} du$$

$$\Rightarrow I = \int u^{-7} du$$

$$\Rightarrow I = -\frac{1}{6}u^{-6} + C$$

$$\Rightarrow I = -\frac{1}{6}(\sin x)^{-6} + C \text{ or } -\frac{1}{6\sin^6 x} + C$$

10      Let  $I = \int_1^2 x^6 \ln x \ dx.$

$$u = \ln x \Rightarrow u' = \frac{1}{x}$$

$$v' = x^6 \Rightarrow v = \frac{1}{7} x^7$$

$$\therefore I = [uv']_1^2 - \int_1^2 u'v \, dx$$

$$\Rightarrow I = \left[ \frac{1}{7} x^7 \ln x \right]_1^2 - \frac{1}{7} \int_1^2 x^6 \, dx$$

$$\Rightarrow I = \left[ \frac{1}{7} x^7 \ln x - \frac{1}{49} x^7 \right]_1^2$$

$$\Rightarrow I = \left[ \frac{x^7}{49} (7 \ln x - 1) \right]_1^2$$

$$\Rightarrow I = \left( \frac{2^7}{49} (7 \ln 2 - 1) \right) - \left( \frac{1^7}{49} (7 \ln 1 - 1) \right)$$

$$\Rightarrow I = \frac{128}{49} \ln 2 - \frac{128}{49} + \frac{1}{49}$$

$$\Rightarrow I = \frac{128}{49} \ln 2 - \frac{127}{49}$$

### Methods in Algebra and Calculus Assessment Standard 1.4

$$11 \quad \frac{dy}{dx} = \frac{3y}{x-5}$$

$$\therefore \int \frac{1}{y} dy = 3 \int \frac{1}{x-5} dx$$

$$\Rightarrow \ln|y| = 3 \ln|x-5| + C$$

$$\Rightarrow y = e^{3 \ln|x-5| + C}$$

$$\Rightarrow y = e^{\ln|x-5|^3} \cdot e^C$$

$$\Rightarrow y = A(x-5)^3 \text{ where } A = e^C$$

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$$\frac{dy}{dx} + 3y = 7e^{6x}$$

$$\text{IF} = e^{\int 3 dx} = e^{3x+C} \Rightarrow \text{IF} = A e^{3x} \text{ where } A = e^C$$

$$\therefore A e^{3x} \frac{dy}{dx} + 3A e^{3x} y = 7A e^{6x} e^{3x}$$

$$\Rightarrow e^{3x} \frac{dy}{dx} + 3e^{3x} y = 7e^{9x}$$

$$\Rightarrow \frac{d}{dx} (e^{3x} y) = 7e^{9x}$$

$$\Rightarrow e^{3x} y = 7 \int e^{9x} dx$$

$$\Rightarrow e^{3x} y = \frac{7}{9} e^{9x} + D$$

$$\Rightarrow y = \frac{7}{9} e^{6x} + D e^{-3x}$$

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$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} - 12y = 0$$

Solving the Auxiliary Equation gives,

$$m^2 + 4m - 12 = 0$$

$$\Rightarrow (m + 6)(m - 2) = 0$$

$$\Rightarrow m = -6, 2$$

$$\therefore y_{\text{GS}}(x) = A e^{-6x} + B e^{2x}$$

$$\therefore y'_{\text{GS}}(x) = -6A e^{-6x} + 2B e^{2x}$$

$$y_{\text{GS}}(0) = 1 \Rightarrow A + B = 1$$

$$y'_{\text{GS}}(0) = 18 \Rightarrow -6A + 2B = 18$$

Hence,

$$\begin{aligned} 6A + 6B &= 6 \\ -6A + 2B &= 18 \end{aligned}$$

$$\therefore 8B = 24 \Rightarrow B = 3$$

$$\therefore A + 3 = 1 \Rightarrow A = -2$$

$$\therefore y_{\text{PS}}(x) = -2e^{-6x} + 3e^{2x}$$