

AH Mathematics

Methods in Algebra and Calculus

Practice Assessment

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Solutions

Methods in Algebra and Calculus Assessment Standard 1.1

$$1 \quad \frac{x^2 + 20}{x^3 + 5x} = \frac{x^2 + 20}{x(x^2 + 5)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 5}; \text{ hence,}$$

$$x^2 + 20 = A(x^2 + 5) + (Bx + C)x$$

$$x = 0 \Rightarrow 20 = 5A \Rightarrow A = 4$$

$$x^2 \text{ coefficients : } 1 = A + B \Rightarrow B = -3$$

$$x \text{ coefficients : } 0 = C$$

Thus,

$$\frac{x^2 + 20}{x^3 + 5x} = \frac{4}{x} - \frac{3x}{x^2 + 5}$$

Methods in Algebra and Calculus Assessment Standard 1.2

$$2 \quad f(x) = e^{x^2 + 9x}$$

$$\Rightarrow f'(x) = e^{x^2 + 9x} \cdot \frac{d}{dx} e^{x^2 + 9x}$$

$$\Rightarrow f'(x) = (2x + 9)e^{x^2 + 9x}$$

$$3 \quad y = \sqrt{\tan 4x} = (\tan 4x)^{\frac{1}{2}}; \text{ hence,}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} (\tan 4x)^{-\frac{1}{2}} \cdot \frac{d}{dx} \tan 4x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} (\tan 4x)^{-\frac{1}{2}} \cdot 4 \sec^2 4x$$

$$\Rightarrow \frac{dy}{dx} = 2 \sec^2 4x \cdot (\tan 4x)^{-\frac{1}{2}}$$

$$4 \quad \text{a}) \quad f(x) = 2x^4 \cos x$$

$$\therefore f'(x) = 2 \cdot 4x^3 \cdot \cos x + 2 \cdot x^4 \cdot (-\sin x)$$

$$\Rightarrow f'(x) = 8x^3 \cos x - 2x^4 \sin x \text{ or } 2x^3(4 \cos x - x \sin x)$$

$$\text{b)} \quad g(x) = \frac{2x + 1}{x - 3}$$

$$\therefore g'(x) = \frac{2(x - 3) - 1(2x + 1)}{(x - 3)^2}$$

$$\Rightarrow g'(x) = \frac{2x - 6 - 2x - 1}{(x - 3)^2}$$

$$\Rightarrow g'(x) = -\frac{7}{(x - 3)^2}$$

$$5 \quad f(x) = \cos^{-1}(5x)$$

$$\therefore f'(x) = -\frac{1}{\sqrt{1 - (5x)^2}} \cdot \frac{d}{dx} 5x$$

$$\Rightarrow f'(x) = -\frac{5}{\sqrt{1 - 25x^2}}$$

$$6 \quad x^4 y + xy^3 = 8$$

$$\therefore 4x^3 \cdot y + x^4 \cdot y' + 1 \cdot y^3 + x \cdot 3y^2 y' = 0$$

$$\Rightarrow 4x^3 y + y^3 + x^4 y' + 3xy^2 y' = 0$$

$$\Rightarrow y'(x^4 + 3xy^2) = -(4x^3 y + y^3)$$

$$\Rightarrow y' = -\frac{4x^3 y + y^3}{x^4 + 3xy^2} \quad \text{or} \quad -\frac{y(4x^3 + y^2)}{x(x^3 + 3y^2)}$$

$$7 \quad x = 6t \Rightarrow \dot{x} = 6; \quad y = 14t - 2t^2 \Rightarrow \dot{y} = 14 - 4t. \quad \text{When } t = 1,$$

$$\dot{x} = 6 \quad \text{and} \quad \dot{y} = 10.$$

$$\text{Speed} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\therefore \text{Speed} = \sqrt{6^2 + 10^2}$$

$$\Rightarrow \text{Speed} = \sqrt{136}$$

Methods in Algebra and Calculus Assessment Standard 1.3

8 a) Let $I = \int \frac{4}{\sqrt{1 - (5x)^2}} dx$. The substitution $u = 5x$ gives,

$$u = 5x \Rightarrow \frac{du}{dx} = 5 \Rightarrow dx = \frac{1}{5} du$$

$$\therefore I = \frac{4}{5} \int \frac{1}{\sqrt{1 - u^2}} du$$

$$\Rightarrow I = \frac{4}{5} \sin^{-1} u + C$$

$$\Rightarrow I = \frac{4}{5} \sin^{-1}(5x) + C$$

b) Let $I = \int \frac{4}{8x - 1} dx$. The substitution $u = 8x - 1$ gives,

$$u = 8x - 1 \Rightarrow \frac{du}{dx} = 8 \Rightarrow dx = \frac{1}{8} du$$

$$\therefore I = \frac{1}{8} \int \frac{1}{u} du$$

$$\Rightarrow I = \frac{1}{8} \ln|u| + C$$

$$\Rightarrow I = \frac{1}{8} \ln|8x - 1| + C$$

c) Let $I = \int_0^{\frac{\pi}{18}} \sec^2 3x dx$. The substitution $u = 3x$ gives,

$$u = 3x \Rightarrow \frac{du}{dx} = 3 \Rightarrow dx = \frac{1}{3} du$$

$$x = 0 \Rightarrow u = 0; x = \frac{\pi}{18} \Rightarrow u = \frac{\pi}{6}$$

$$\therefore I = \frac{1}{3} \int_0^{\frac{\pi}{6}} \sec^2 u \, du$$

$$\Rightarrow I = \frac{1}{3} [\tan u]_0^{\frac{\pi}{6}}$$

$$\Rightarrow I = \frac{1}{3} \left(\tan \frac{\pi}{6} - \tan 0 \right)$$

$$\Rightarrow I = \frac{1}{3\sqrt{3}}$$

9 Let $I = \int \frac{\sin x}{\cos^5 x} dx.$

$$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow du = -\sin x \, dx$$

$$\therefore I = - \int \frac{1}{u^5} \, du$$

$$\Rightarrow I = - \int u^{-5} \, du$$

$$\Rightarrow I = \frac{1}{4} u^{-4} + C$$

$$\Rightarrow I = \frac{1}{4} (\cos x)^{-4} + C \quad \text{or} \quad \frac{1}{4 \cos^4 x} + C$$

10 Let $I = \int_1^2 x^4 \ln x \, dx.$

$$u = \ln x \Rightarrow u' = \frac{1}{x}$$

$$v' = x^4 \Rightarrow v = \frac{1}{5} x^5$$

$$\therefore I = [uv']_1^2 - \int_1^2 u'v \, dx$$

$$\Rightarrow I = \left[\frac{1}{5} x^5 \ln x \right]_1^2 - \frac{1}{5} \int_1^2 x^4 \, dx$$

$$\Rightarrow I = \left[\frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 \right]_1^2$$

$$\Rightarrow I = \left[\frac{x^5}{25} (5 \ln x - 1) \right]_1^2$$

$$\Rightarrow I = \left(\frac{2^5}{25} (5 \ln 2 - 1) \right) - \left(\frac{1^6}{25} (5 \ln 1 - 1) \right)$$

$$\Rightarrow I = \frac{32}{5} \ln 2 - \frac{32}{25} + \frac{1}{25}$$

$$\Rightarrow I = \frac{32}{5} \ln 2 - \frac{31}{25}$$

Methods in Algebra and Calculus Assessment Standard 1.4

$$11 \quad \frac{dy}{dx} = \frac{2y}{x-7}$$

$$\therefore \int \frac{1}{y} dy = 2 \int \frac{1}{x-7} dx$$

$$\Rightarrow \ln|y| = 2 \ln|x-7| + C$$

$$\Rightarrow y = e^{2 \ln|x-7| + C}$$

$$\Rightarrow y = e^{\ln|x-7|^2} \cdot e^C$$

$$\Rightarrow y = A(x-7)^2 \text{ where } A = e^C$$

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$$\frac{dy}{dx} + 4y = 3e^x$$

$$\text{IF} = e^{\int 4 dx} = e^{4x+C} \Rightarrow \text{IF} = A e^{4x} \text{ where } A = e^C$$

$$\therefore A e^{4x} \frac{dy}{dx} + 4A e^{4x} y = 3A e^{4x} e^x$$

$$\Rightarrow e^{4x} \frac{dy}{dx} + 4e^{4x} y = 3e^{5x}$$

$$\Rightarrow \frac{d}{dx} (e^{4x} y) = 3e^{5x}$$

$$\Rightarrow e^{4x} y = 3 \int e^{5x} dx$$

$$\Rightarrow e^{4x} y = \frac{3}{5} e^{5x} + D$$

$$\Rightarrow y = \frac{4}{5} e^{3x} + D e^{-4x}$$

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$$\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} - 4y = 0$$

Solving the Auxiliary Equation gives,

$$m^2 + 3m - 4 = 0$$

$$\Rightarrow (m+4)(m-1) = 0$$

$$\Rightarrow m = -4, 1$$

$$\therefore y_{\text{GS}}(x) = A e^{-4x} + B e^x$$

$$\therefore y'_{\text{GS}}(x) = -4A e^{-4x} + B e^x$$

$$y_{\text{GS}}(0) = 2 \Rightarrow A + B = 2$$

$$y'_{\text{GS}}(0) = 7 \Rightarrow -4A + B = 7$$

Hence,

$$\therefore 5A = -5 \Rightarrow A = -1$$

$$\therefore -1 + B = 2 \Rightarrow B = 3$$

$$\therefore y_{\text{PS}}(x) = -e^{-4x} + 3e^x$$