

AH Mathematics

Methods in Algebra and Calculus

Practice Assessment

1

Solutions

Methods in Algebra and Calculus Assessment Standard 1.1

$$1 \quad \frac{x^2 + 12}{x^3 + 6x} = \frac{x^2 + 12}{x(x^2 + 6)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 6}; \text{ hence,}$$

$$x^2 + 12 = A(x^2 + 6) + (Bx + C)x$$

$$x = 0 \Rightarrow 12 = 6A \Rightarrow A = 2$$

$$x^2 \text{ coefficients : } 1 = A + B \Rightarrow B = -1$$

$$x \text{ coefficients : } 0 = C$$

Thus,

$$\frac{x^2 + 12}{x^3 + 6x} = \frac{2}{x} - \frac{x}{x^2 + 6}$$

Methods in Algebra and Calculus Assessment Standard 1.2

$$2 \quad f(x) = e^{x^2 - 5x}$$

$$\Rightarrow f'(x) = e^{x^2 - 5x} \cdot \frac{d}{dx} e^{x^2 - 5x}$$

$$\Rightarrow f'(x) = (2x - 5)e^{x^2 - 5x}$$

$$3 \quad y = \sqrt{\tan 3x} = (\tan 3x)^{\frac{1}{2}}; \text{ hence,}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} (\tan 3x)^{-\frac{1}{2}} \cdot \frac{d}{dx} \tan 3x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} (\tan 3x)^{-\frac{1}{2}} \cdot 3 \sec^2 3x$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2} \sec^2 3x \cdot (\tan 3x)^{-\frac{1}{2}} \text{ or } \frac{3 \sec^2 3x}{2 \sqrt{\tan 3x}}$$

$$4 \quad \text{a) } f(x) = 6x^3 \sin x$$

$$\therefore f'(x) = 6 \cdot 3x^2 \cdot \sin x + 6 \cdot x^3 \cdot \cos x$$

$$\Rightarrow f'(x) = 18x^2 \sin x + 6x^3 \cos x \text{ or } 6x^2(3 \sin x + x \cos x)$$

$$\text{b)} \quad g(x) = \frac{4x + 3}{x - 2}$$

$$\therefore g'(x) = \frac{4(x - 2) - 1(4x + 3)}{(x - 2)^2}$$

$$\Rightarrow g'(x) = \frac{4x - 8 - 4x - 3}{(x - 2)^2}$$

$$\Rightarrow g'(x) = -\frac{11}{(x - 2)^2}$$

$$5 \quad f(x) = \sin^{-1}(4x)$$

$$\therefore f'(x) = \frac{1}{\sqrt{1 - (4x)^2}} \cdot \frac{d}{dx} 4x$$

$$\Rightarrow f'(x) = \frac{4}{\sqrt{1 - 16x^2}}$$

$$6 \quad x^3 y + x y^4 = 13$$

$$\therefore 3x^2 \cdot y + x^3 \cdot y' + 1 \cdot y^4 + x \cdot 4y^3 y' = 0$$

$$\Rightarrow 3x^2 y + y^4 + x^3 y' + 4x y^3 y' = 0$$

$$\Rightarrow y'(x^3 + 4x y^3) = -(3x^2 y + y^4)$$

$$\Rightarrow y' = -\frac{3x^2 y + y^4}{x^3 + 4x y^3} \quad \text{or} \quad -\frac{y(3x^2 + y^3)}{x(x^2 + 4y^3)}$$

$$7 \quad x = 7t \Rightarrow \dot{x} = 7; \quad y = 16t - 2t^2 \Rightarrow \dot{y} = 16 - 4t. \quad \text{When } t = 3,$$

$$\dot{x} = 7 \quad \text{and} \quad \dot{y} = 4.$$

$$\text{Speed} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\therefore \text{Speed} = \sqrt{7^2 + 4^2}$$

$$\Rightarrow \text{Speed} = \sqrt{65}$$

Methods in Algebra and Calculus Assessment Standard 1.3

8 a) Let $I = \int \frac{7}{\sqrt{1 - (4x)^2}} dx$. The substitution $u = 4x$ gives,

$$u = 4x \Rightarrow \frac{du}{dx} = 4 \Rightarrow dx = \frac{1}{4} du$$

$$\therefore I = \frac{7}{4} \int \frac{1}{\sqrt{1 - u^2}} du$$

$$\Rightarrow I = \frac{7}{4} \sin^{-1} u + C$$

$$\Rightarrow I = \frac{7}{4} \sin^{-1}(4x) + C$$

b) Let $I = \int \frac{3}{6x - 5} dx$. The substitution $u = 6x - 5$ gives,

$$u = 6x - 5 \Rightarrow \frac{du}{dx} = 6 \Rightarrow dx = \frac{1}{6} du$$

$$\therefore I = \frac{1}{6} \int \frac{1}{u} du$$

$$\Rightarrow I = \frac{1}{6} \ln|u| + C$$

$$\Rightarrow I = \frac{1}{6} \ln|6x - 5| + C$$

c) Let $I = \int_0^{\frac{\pi}{24}} \sec^2 8x dx$. The substitution $u = 8x$ gives,

$$u = 8x \Rightarrow \frac{du}{dx} = 8 \Rightarrow dx = \frac{1}{8} du$$

$$x = 0 \Rightarrow u = 0; x = \frac{\pi}{24} \Rightarrow u = \frac{\pi}{3}$$

$$\therefore I = \frac{1}{8} \int_0^{\frac{\pi}{3}} \sec^2 u \, du$$

$$\Rightarrow I = \frac{1}{8} [\tan u]_0^{\frac{\pi}{3}}$$

$$\Rightarrow I = \frac{1}{8} \left(\tan \frac{\pi}{3} - \tan 0 \right)$$

$$\Rightarrow I = \frac{\sqrt{3}}{8}$$

9 Let $I = \int \frac{\sin x}{\cos^3 x} dx.$

$$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow du = -\sin x \, dx$$

$$\therefore I = - \int \frac{1}{u^3} \, du$$

$$\Rightarrow I = - \int u^{-3} \, du$$

$$\Rightarrow I = - \frac{1}{2} u^{-2} + C$$

$$\Rightarrow I = \frac{1}{2} (\cos x)^{-2} + C \text{ or } \frac{1}{2 \cos^2 x} + C$$

10 Let $I = \int_1^2 x^5 \ln x \, dx.$

$$u = \ln x \Rightarrow u' = \frac{1}{x}$$

$$v' = x^5 \Rightarrow v = \frac{1}{6} x^6$$

$$\therefore I = [uv']_1^2 - \int_1^2 u'v \, dx$$

$$\Rightarrow I = \left[\frac{1}{6} x^6 \ln x \right]_1^2 - \frac{1}{6} \int_1^2 x^5 \, dx$$

$$\Rightarrow I = \left[\frac{1}{6} x^6 \ln x - \frac{1}{36} x^6 \right]_1^2$$

$$\Rightarrow I = \left[\frac{x^6}{36} (6 \ln x - 1) \right]_1^2$$

$$\Rightarrow I = \left(\frac{2^6}{36} (6 \ln 2 - 1) \right) - \left(\frac{1^6}{36} (6 \ln 1 - 1) \right)$$

$$\Rightarrow I = \frac{64}{6} \ln 2 - \frac{64}{36} + \frac{1}{36}$$

$$\Rightarrow I = \frac{32}{3} \ln 2 - \frac{7}{4}$$

Methods in Algebra and Calculus Assessment Standard 1.4

$$11 \quad \frac{dy}{dx} = \frac{4y}{x-1}$$

$$\therefore \int \frac{1}{y} dy = 4 \int \frac{1}{x-1} dx$$

$$\Rightarrow \ln|y| = 4 \ln|x-1| + C$$

$$\Rightarrow y = e^{4 \ln|x-1| + C}$$

$$\Rightarrow y = e^{\ln|x-1|^4} \cdot e^C$$

$$\Rightarrow y = A(x-1)^4 \text{ where } A = e^C$$

12

$$\frac{dy}{dx} + 2y = 4e^{3x}$$

$$\text{IF} = e^{\int 2 dx} = e^{2x+C} \Rightarrow \text{IF} = A e^{2x} \text{ where } A = e^C$$

$$\therefore A e^{2x} \frac{dy}{dx} + 2A e^{2x} y = 4A e^{2x} e^{3x}$$

$$\Rightarrow e^{2x} \frac{dy}{dx} + 2e^{2x} y = 4e^{5x}$$

$$\Rightarrow \frac{d}{dx} (e^{2x} y) = 4e^{5x}$$

$$\Rightarrow e^{2x} y = 4 \int e^{5x} dx$$

$$\Rightarrow e^{2x} y = \frac{4}{5} e^{5x} + D$$

$$\Rightarrow y = \frac{4}{5} e^{3x} + D e^{-2x}$$

13

$$\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 15y = 0$$

Solving the Auxiliary Equation gives,

$$m^2 + 2m - 15 = 0$$

$$\Rightarrow (m+5)(m-3) = 0$$

$$\Rightarrow m = -5, 3$$

$$\therefore y_{\text{GS}}(x) = A e^{-5x} + B e^{3x}$$

$$\therefore y'_{\text{GS}}(x) = -5A e^{-5x} + 3B e^{3x}$$

$$y_{\text{GS}}(0) = 10 \Rightarrow A + B = 10$$

$$y'_{\text{GS}}(0) = 2 \Rightarrow -5A + 3B = 2$$

Hence,

$$\begin{array}{rcl} 5A & + & 5B = 50 \\ -5A & + & 3B = 2 \end{array}$$

$$\therefore 8B = 52 \Rightarrow B = \frac{13}{2}$$

$$\therefore A + \frac{13}{2} = \frac{20}{2} \Rightarrow A = \frac{7}{2}$$

$$\therefore y_{\text{PS}}(x) = \frac{7}{2}e^{-5x} + \frac{13}{2}e^{3x}$$