Perth Academy

Mathematics Department

Higher

Key Points

Wave Function

Wave Function

1 $a \cos x + b \sin x$ can be written in the forms:

- $k \cos (x - \alpha)$
- $k \cos (x + \alpha)$
- $k \sin (x - \alpha)$
- $k \sin (x + \alpha)$

where $k = \sqrt{a^2 + b^2}$ and $\tan \alpha$ can be calculated using a and b. You must first expand $\cos (x \pm \alpha)$ or $\sin (x \pm \alpha)$ and then equate coefficients.

2 The maximum and minimum values of $a \cos x + b \sin x$ are given by the maximum and minimum values of any of:

- $k \cos (x - \alpha)$
- $k \cos (x + \alpha)$
- $k \sin (x - \alpha)$
- $k \sin (x + \alpha)$

3 The solutions of the equation $a \cos x + b \sin x = c$ can be obtained from any of the equations:

- $k \cos (x - \alpha) = c$
- $k \cos (x + \alpha) = c$
- $k \sin (x - \alpha) = c$
- $k \sin (x + \alpha) = c$

Example 1

Express $2 \sin x° - 5 \cos x°$ in the form $k \sin (x - \alpha)°$, where $k > 0$ and $0 \leqslant \alpha \leqslant 360$. [Higher]

Solution

$$\begin{aligned} 2 \sin x° - 5 \cos x° &= k \sin (x - \alpha)° \\ &= k (\sin x° \cos \alpha° - \cos x° \sin \alpha°) \\ &= k \cos \alpha° \sin x° - k \sin \alpha° \cos x° \end{aligned}$$

Hence $k \cos \alpha° = 2$

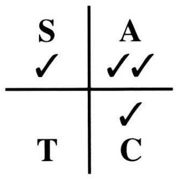
$k \sin \alpha° = 5$

$k = \sqrt{2^2 + 5^2} = \sqrt{29}$

$\tan \alpha° = \frac{5}{2}$

$\alpha = 68.2$

$\alpha°$ is in the first quadrant as both $\cos \alpha°$ and $\sin \alpha°$ are positive



$2 \sin x° - 5 \cos x° = \sqrt{29} \sin(x - 68.2)°$

Example 2

Express $3 \cos x° - 4 \sin x°$ in the form $k \cos(x - \alpha)°$ where $k > 0$ and $0 \leqslant \alpha \leqslant 360$.

Solution

$$
\begin{aligned}
3 \cos x° - 4 \sin x° &= k \cos (x - \alpha)° \\
&= k (\cos x° \cos \alpha° + \sin x° \sin \alpha°) \\
&= k \cos \alpha° \cos x° + k \sin \alpha° \sin x°
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence } k \cos \alpha° &= 3 \\
k \sin \alpha° &= -4 \\
k &= \sqrt{3^2 + (-4)^2 = 5} \\
\tan \alpha° &= \frac{-4}{3} \\
\alpha &= 306.9
\end{aligned}
$$

α is in the fourth quadrant as $\cos \alpha°$ is positive and $\sin \alpha°$ is negative

S	A ✓
✓ T	✓✓ C

$$3 \cos x° - 4 \sin x° = 5 \cos (x - 306.9)°$$

Example 3

Express $\sqrt{3} \cos \theta - \sin \theta$ in the form $r \sin (\theta + \alpha)$ where $r > 0$ and $0 \leqslant \alpha \leqslant 2\pi$.

Solution

$$
\begin{aligned}
\sqrt{3} \cos \theta - \sin \theta &= r \sin (\theta + \alpha) \\
&= r (\sin \theta \cos \alpha + \cos \theta \sin \alpha) \\
&= r \cos \alpha \sin \theta + r \sin \alpha \cos \theta
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence } r \cos \alpha &= -1 \\
r \sin \alpha &= \sqrt{3} \\
r &= \sqrt{(-1)^2 + \sqrt{3}^2} = 2 \\
\tan \alpha &= \frac{\sqrt{3}}{-1} \\
\alpha &= \frac{2\pi}{3}
\end{aligned}
$$

α is in the second quadrant as $\cos \alpha$ is negative and $\sin \alpha$ is positive

S ✓✓	A ✓
✓ T	C

$$\sqrt{3} \cos \theta - \sin \theta = 2 \sin \left(\theta + \frac{2\pi}{3}\right)$$

Example 4

When two sound waves are added together the volume, V, at any time, t seconds, is given by $V(t) = 40 \cos t° + 20 \sin t°$. Find the maximum volume and the time t at which this maximum first occurs.

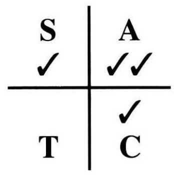
Solution

$$
\begin{aligned}
40 \cos t° + 20 \sin t° &= k \cos (t - \alpha)° \\
&= k (\cos t° \cos \alpha° + \sin t° \sin \alpha°) \\
&= k \cos \alpha° \cos t° + k \sin \alpha° \sin t°
\end{aligned}
$$

Hence

$$
\begin{aligned}
k \cos \alpha° &= 40 \\
k \sin \alpha° &= 20 \\
k &= \sqrt{40^2 + 20^2} = 20\sqrt{5} \\
\tan \alpha° &= \frac{20}{40} \\
\alpha &= 26.6
\end{aligned}
$$

α is in the first quadrant as both $\cos \alpha°$ and $\sin \alpha°$ are positive



$$
\begin{aligned}
V(t) &= 40 \cos t° + 20 \sin t° \\
&= 20\sqrt{5} \cos (t - 26.6)°
\end{aligned}
$$

The maximum value of $\cos (t - 26.6)°$ is 1

Hence the maximum value of $20\sqrt{5} \cos (t - 26.6)°$ is $20\sqrt{5}$

This maximum occurs when $\cos (t - 26.6)° = 1$

so $(t - 26.6) = 0$

$t = 26.6$

The maximum value is $20\sqrt{5}$ and first occurs at 26.6 seconds.

Example 5

Solve algebraically $\sqrt{2}\sin\theta - \sqrt{6}\cos\theta = 2$ for $0 \leqslant \theta \leqslant 2\pi$.

Solution

$$\begin{aligned}\sqrt{2}\sin\theta - \sqrt{6}\cos\theta &= k\cos(\theta - \alpha)\\ &= k(\cos\theta\cos\alpha + \sin\theta\sin\alpha)\\ &= k\cos\alpha\cos\theta + k\sin\alpha\sin\theta\end{aligned}$$

Hence $k\cos\alpha = -\sqrt{6}$

$k\sin\alpha = \sqrt{2}$

$$k = \sqrt{(\sqrt{2})^2 + (-\sqrt{6})^2} = 2\sqrt{2}$$

$$\tan\alpha = \frac{\sqrt{2}}{-\sqrt{6}} = \frac{-1}{\sqrt{3}}$$

$$\alpha = \frac{5\pi}{6}$$

α is in the second quadrant as $\cos\alpha$ is negative and $\sin\alpha$ is positive

S	A
✓✓	✓
✓	
T	C

$$\sqrt{2}\sin\theta - \sqrt{6}\cos\theta = 2$$

$$2\sqrt{2}\cos\left(\theta - \frac{5\pi}{6}\right) = 2$$

$$\cos\left(\theta - \frac{5\pi}{6}\right) = \frac{1}{\sqrt{2}}$$

$$\left(\theta - \frac{5\pi}{6}\right) = \frac{\pi}{4} \text{ or } \frac{-\pi}{4}$$

$\left(\text{Since } 0 \leqslant \theta \leqslant 2\pi,\ -\frac{5\pi}{6} \leqslant \theta - \frac{5\pi}{6} \leqslant \frac{7\pi}{6}\right)$

$$\theta = \frac{\pi}{4} + \frac{5\pi}{6} \text{ or } \frac{-\pi}{4} + \frac{5\pi}{6}$$

$$\theta = \frac{13\pi}{12} \text{ or } \frac{7\pi}{12}$$

Example 6

A research student observing waves in a tank uses the formula $h = 1.5 + 2\cos 30t° - \sin 30t°$, where h is the height of the wave in metres and t is the time in seconds after the start of the experiment. The wave may overflow the tank if its height exceeds 3.5 metres.
Between which times is the wave first in danger of overflowing?

Solution

$$2\cos 30t° - \sin 30t° = k\cos(30t - \alpha)°$$
$$= k(\cos 30t° \cos\alpha° + \sin 30t° \sin\alpha°)$$
$$= k\cos\alpha° \cos 30t° + k\sin\alpha° \sin 30t°$$

Hence $k\cos\alpha° = 2$

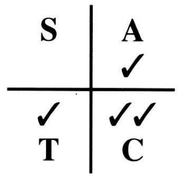
$k\sin\alpha° = -1$

$k = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$\tan\alpha° = \frac{-1}{2}$

$\alpha = 333$

α is in the fourth quadrant as $\cos\alpha°$ is positive and $\sin\alpha°$ is negative



$2\cos 30t° - \sin 30t° = \sqrt{5}\cos(30t - 333)°$

Since the wave tank is full when $h = 3.5$

$1.5 + 2\cos 30t° - \sin 30t° = 3.5$

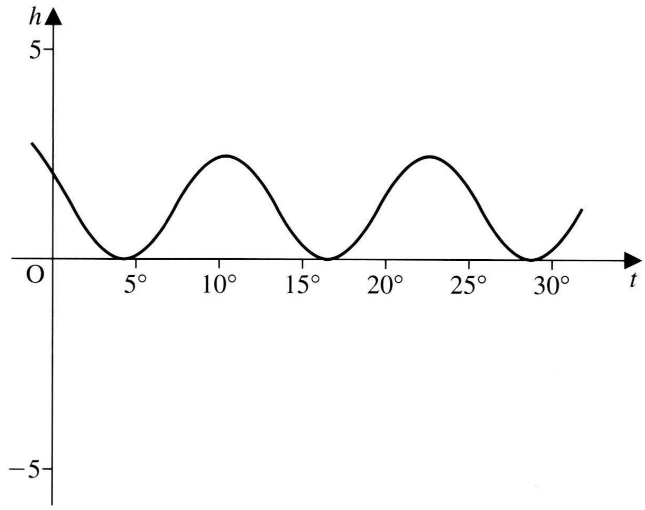
$\sqrt{5}\cos(30t - 333)° = 2$

$\cos(30t - 333)° = 0.894$

$30t - 333 = -26.6$ or 26.6

$30t = 306.4$ or 359.6

$t = 10.1$ or 11.9



The wave is first in danger of overflowing between 10.1 and 11.9 seconds after the start of the experiment.