Perth Academy

Mathematics Department

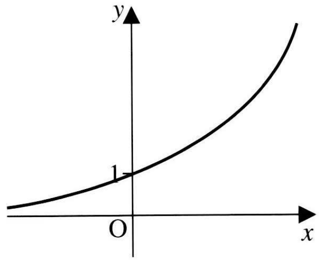
Higher

Key Points

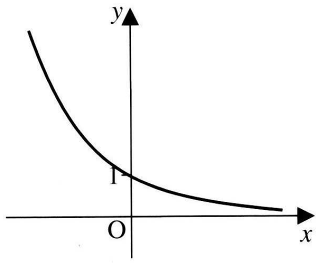
Exponential and Log Functions

Exponential and Log Functions

1 For $y = a^x$, $a > 1$, the graph
- is always positive
- never crosses the x-axis
- is increasing
- passes through (0, 1).

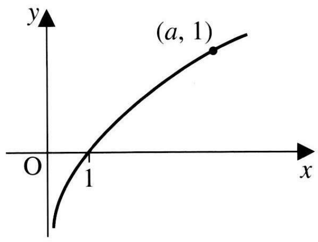


2 For $y = a^x$, $0 < a < 1$, the graph
- is always positive
- never crosses the x-axis
- is decreasing
- passes through (0, 1).



3 A function of the form $y = a^x$ is called an **exponential function** to the base a, $a =\neq 0$.
$f(x) = 2.718^x = e^x$ is called the exponential function to the base e.

4 The graph of $y = \log_a x$
- cuts the x-axis at (1, 0)
- passes through (a, 1).



5 If $y = a^x$ then $x = \log_a y$
If $y = \log_a x$ then $x = a^y$

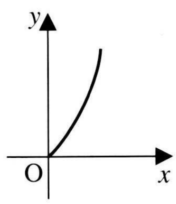
6 $\log_a xy = \log_a x + \log_a y$

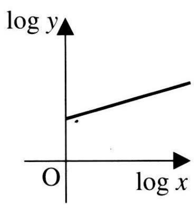
7 $\log_a \frac{x}{y} = \log_a x - \log_a y$

8 $\log_a x^n = n \log_a x$

9 $\log_e x = \ln x$

10 If $y = kx^n$ then $\log y = n \log x + \log k$.
This is a linear equation. Hence, if the graph of $\log y$ against $\log x$ is a straight line, then the formula is of type $y = kx^n$. The straight line graph may be used to determine k and n.





Example 1

Simplify

(a) $\log_4 64$

(b) $\log_4 16 + \log_4 8 - \log_4 32$

(c) $\frac{1}{3}\log_9 27$

Solution

(a) Since $4^3 = 64$

$$\log_4 64 = \log_4 4^3 = 3$$

(b)
$$\begin{aligned}\log_4 16 + \log_4 8 - \log_4 32 &= \log_4\left(\frac{16 \times 8}{32}\right)\\ &= \log_4 4\\ &= 1\end{aligned}$$

(c)
$$\begin{aligned}\tfrac{1}{3}\log_9 27 &= \log_9\left(27^{\frac{1}{3}}\right)\\ &= \log_9 3\\ &= \log_9 9^{\frac{1}{2}}\\ &= \tfrac{1}{2}\end{aligned}$$

Example 2

Given $3^x = 10$, find an expression for the exact value of x.

Solution

$$\begin{aligned}3^x &= 10\\ \log_{10}(3^x) &= \log_{10} 10\\ x\log_{10} 3 &= 1\\ x &= \frac{1}{\log_{10} 3}\end{aligned}$$

Example 3

Given that $\log_{10} y = 3.4$, write down an expression for the exact value of y.

Solution

$y = 10^{3.4}$

Example 4

Solve, for $x > 0$, $\log_7(x^2 - 1) - \log_7(x - 1) = 2$.

Solution

$\log_7(x^2 - 1) - \log_7(x - 1) = 2$

$$\begin{aligned}\log_7\frac{(x^2 - 1)}{(x - 1)} &= \log_7 49\\ \frac{x^2 - 1}{x - 1} &= 49\\ \frac{(x + 1)(x - 1)}{x - 1} &= 49\\ x + 1 &= 49\\ x &= 48\end{aligned}$$

notice that
$\log_7 49 = \log_7 7^2 = 2$

Example 5

The air pressure in a life raft falls according to the formula $P_t = P_0 e^{-kt}$, where P_0 is the initial pressure, P_t is the pressure at time t hours and k is a constant.

(a) At time zero the pressure is 80 units. 12 hours later it is 60 units. Find the value of k to two significant figures.

(b) When the pressure is below 40 units the raft is unsafe. From time zero, for how long is the raft safe to use?

Solution

(a)
$$\begin{aligned} P_t &= P_0 e^{-kt} \\ 60 &= 80e^{-k(12)} \\ e^{-12k} &= 0.75 \\ \ln e^{-12k} &= \ln 0.75 \\ -12k &= -0.288 \\ k &= 0.02 \end{aligned}$$

(b)
$$\begin{aligned} P_t &= P_0 e^{-0.02t} \\ 40 &= 80e^{-0.02t} \\ e^{-0.02t} &= 0.5 \\ \ln e^{-0.02t} &= \ln 0.5 \\ -0.02t &= -0.693 \\ t &= 34.7\text{ h or } 34\text{ h } 42\text{ min} \end{aligned}$$

Example 6

From the experimental data given in the table:

(a) show that x and y are related by the formula $y = kx^n$

(b) find the value of k and n, and state the formula that connects x and y.

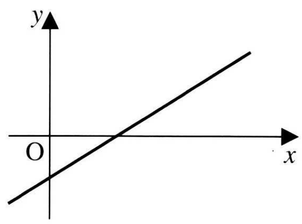
x	10.1	19.9	31	39.5
y	2.5	4.1	5.5	6.6

Solution

(a) Taking logarithms to base 10 of x and y gives:

$\log_{10} x$	1.0	1.3	1.5	1.6
$\log_{10} y$	0.40	0.61	0.74	0.82

These points lie on a straight line. Hence, the formula connecting x and y is of the form $y = kx^n$.



(b) Taking two points on the line $Y = nX + C$, $Y = \log_{10} y$, $X = \log_{10} x$ and $C = \log_{10} k$.

$Y = nX + C$

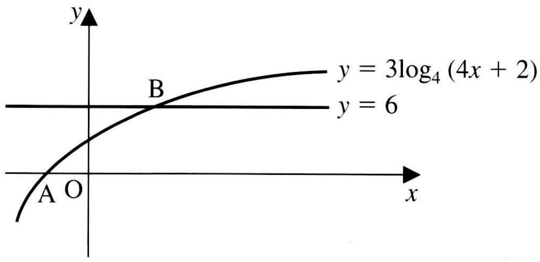
$0.61 = n1.3 + C$

$0.82 = n1.6 + C$

Solving simultaneously gives $n = 0.7$ and $C = -0.3$.
Hence $k = 0.5$ and $y = 0.5x^{0.7}$

Example 7

Part of the graph of $y = 3\log_4(4x + 2)$ is shown in the diagram. The graph crosses the x-axis at the point A and crosses the straight line $y = 6$ at the point B.
Find the x-coordinate of B.



Solution

$y = 3\log_4(4x + 2)$

At $y = 6$, $\quad y = 3\log_4(4x + 2) = 6$

$$\log_4(4x + 2) = 2$$
$$4x + 2 = 4^2$$
$$4x = 14$$
$$x = \frac{7}{2}$$