Perth Academy

Mathematics Department

Higher

Key Points

Vectors

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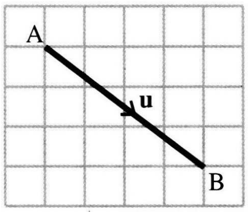
1 A **vector** is a quantity with both magnitude (size) and direction. A vector is named using either the letters at the end of the directed line segment $\overrightarrow{AB}$ or a bold letter **u**. A vector may also be represented by its **components**. These are known as **column vectors**.

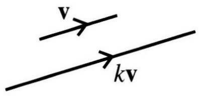
$\overrightarrow{AB} = \mathbf{u} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$



2 If $\overrightarrow{PQ} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then $|\overrightarrow{PQ}| = \sqrt{a^2 + b^2 + c^2}$

3 If vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $k\mathbf{v} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$ and vector $k\mathbf{v}$ is parallel to vector **v**.

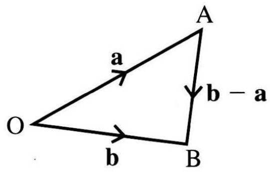
Hence if $\mathbf{u} = k\mathbf{v}$ then **u** is parallel to **v**.
Conversely if **u** is parallel to **v** then $\mathbf{u} = k\mathbf{v}$.



4 $\overrightarrow{OA}$ is called the **position vector** of the point A relative to origin O, written **a**.

$\overrightarrow{OB}$ is called the position vector of B, written **b**.

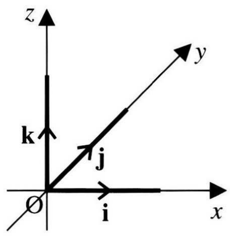
$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ where **a** and **b** are the position vectors of A and B.



5 Points are said to be **collinear** if they lie on a straight line. If $\overrightarrow{AB} = k\overrightarrow{BC}$, where k is a scalar, then $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$. If B is also a point common to both $\overrightarrow{AB}$ and $\overrightarrow{BC}$ then A, B and C are collinear.

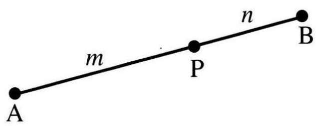
6 A vector may also be defined in terms of **i**, **j** and **k**, where **i**, **j** and **k** are unit vectors in the x, y and z directions, respectively. In component form these vectors are written as

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



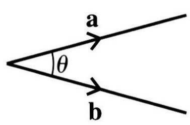
7 If **p** is the position vector of the point P that divides AB in the ratio $m : n$ then

$\mathbf{p} = \dfrac{n}{m+n}\mathbf{a} + \dfrac{m}{m+n}\mathbf{b}$



8 For two vectors **a** and **b** the scalar product is defined as $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, where θ is the angle between **a** and **b**, $0 \leqslant \theta \leqslant 180°$.

9 If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$



10 If **a** and **b** are perpendicular then $\mathbf{a}.\mathbf{b} = 0$.
Conversely if $\mathbf{a}.\mathbf{b} = 0$ then **a** and **b** are perpendicular.

11 $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}||\mathbf{b}|}$ or $\cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

12 For vectors **a** and **b**, $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$

13 For vectors **a**, **b** and **c**, $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$

Example 1

For the points P(−2, 1, 5) and Q(−3, −5, 6) find the components of $\overrightarrow{PQ}$ and calculate its magnitude.

Solution

$$\begin{aligned}\overrightarrow{PQ} &= \mathbf{q} - \mathbf{p}\\ &= \begin{pmatrix}-3\\-5\\6\end{pmatrix} - \begin{pmatrix}-2\\1\\5\end{pmatrix}\\ &= \begin{pmatrix}-1\\-6\\1\end{pmatrix}\end{aligned}$$

$$\begin{aligned}|\overrightarrow{PQ}| &= \sqrt{(-1)^2 + (-6)^2 + 1^2}\\ &= \sqrt{38}\end{aligned}$$

Example 2

A, B and C have coordinates (2, 4, 6), (6, 6, 2) and (14, 10, −6).

(a) Write down the components of $\overrightarrow{AB}$.
(b) Hence show that A, B and C are collinear.
(c) Find the value of AB : BC.

Solution

(a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix}6\\6\\2\end{pmatrix} - \begin{pmatrix}2\\4\\6\end{pmatrix} - \begin{pmatrix}4\\2\\-4\end{pmatrix}$

(b) $\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix}14\\10\\-6\end{pmatrix} - \begin{pmatrix}6\\6\\2\end{pmatrix} - \begin{pmatrix}8\\4\\-8\end{pmatrix}$

$\overrightarrow{BC} = 2\overrightarrow{AB}$ hence $\overrightarrow{BC}$ is parallel to $\overrightarrow{AB}$. Since B is a point in common, A, B and C are collinear.

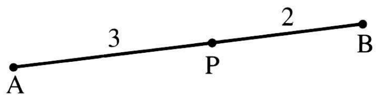
(c) AB : BC = 1 : 2

Example 3

P divides AB in the ratio 3 : 2. If A is the point (−3, 1, 1) and B is (2, 1, −4), find the coordinates of P.

Solution

$$\begin{aligned}\mathbf{p} &= \frac{2\mathbf{a}}{5} + \frac{3\mathbf{b}}{5}\\ &= \frac{1}{5}(2\mathbf{a} + 3\mathbf{b})\\ &= \frac{1}{5}\left[2\begin{pmatrix}-3\\1\\1\end{pmatrix} + 3\begin{pmatrix}2\\1\\-4\end{pmatrix}\right]\\ &= \frac{1}{5}\begin{pmatrix}0\\5\\-10\end{pmatrix} = \begin{pmatrix}0\\1\\-2\end{pmatrix}\end{aligned}$$



P has coordinates (0, 1, −2).

Example 4

Points P, Q and R have coordinates (−1, 0, 3), (2, 3, −1) and (1, 5, −4).

(a) Calculate $\overrightarrow{QP}.\overrightarrow{QR}$

(b) Hence find the size of angle PQR.

Solution

(a) $\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$ $\qquad$ $\overrightarrow{QR} = \mathbf{r} - \mathbf{q}$

$$= \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \qquad = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ -3 \\ 4 \end{pmatrix} \qquad = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

Hence $\overrightarrow{QP}.\overrightarrow{QR} = -3.-1 + -3.2 + 4.-3$
$= -15$

(b) $\cos PQR = \dfrac{\overrightarrow{QP}.\overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|} = \dfrac{-15}{\sqrt{(-3)^2 + (-3)^2 + 4^2} \times \sqrt{(-1)^2 + 2^2 + (-3)^2}}$

$= \dfrac{-15}{\sqrt{34} \times \sqrt{14}}$

$\cos PQR = -0.688$

Hence angle PQR is 133.4°.

Example 5

For vectors $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ p \end{pmatrix}$ find p if **a** is perpendicular to **b**.

Solution

If **a** is perpendicular to **b** then **a.b** = 0

Hence, $2.-1 + -1.1 + -3.p = 0$

$-3 - 3p = 0$

$p = -1$

Example 6

EFGH is a quadrilateral with vertices E(8, −2, 6), F(16, 6, −2), G(0, 8, 8) and H(−8, 0, 16).

(a) Find the coordinates of M, the mid–point of EF.
(b) Find the coordinates of N, which divides GM in the ratio 2 : 1
(c) Find the ratio in which N divides FH.

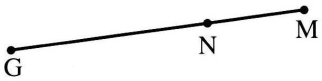
Solution

(a) $\mathbf{m} = \frac{1}{2}(\mathbf{e} + \mathbf{f})$

$$= \frac{1}{2}\begin{pmatrix}24\\4\\4\end{pmatrix} = \begin{pmatrix}12\\2\\2\end{pmatrix}$$

M(12, 2, 2)

(b) $\mathbf{n} = \frac{2}{3}\mathbf{m} + \frac{1}{3}\mathbf{g}$



$$= \frac{1}{3}(2\mathbf{m} + \mathbf{g})$$

$$= \frac{1}{3}\left[\begin{pmatrix}24\\4\\4\end{pmatrix} + \begin{pmatrix}0\\8\\8\end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{pmatrix}24\\12\\12\end{pmatrix} = \begin{pmatrix}8\\4\\4\end{pmatrix}$$

N(8, 4, 4)

(c) $\overrightarrow{FN} = \mathbf{n} - \mathbf{f}$ $\qquad$ $\overrightarrow{NH} = \mathbf{h} - \mathbf{n}$

$$= \begin{pmatrix}8\\4\\4\end{pmatrix} - \begin{pmatrix}16\\6\\-2\end{pmatrix} \qquad = \begin{pmatrix}-8\\0\\16\end{pmatrix} - \begin{pmatrix}8\\4\\4\end{pmatrix}$$

$$= \begin{pmatrix}-8\\-2\\6\end{pmatrix} \qquad = \begin{pmatrix}-16\\-4\\12\end{pmatrix}$$

$\overrightarrow{NH} = 2\overrightarrow{FN}$, hence N divides $\overrightarrow{FH}$ in the ratio 1 : 2.

Example 7

The diagram shows a square-based pyramid. All edges are 2 units in length. Calculate **a**.(**b** + **c**).

Solution

$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$
$$= 2 \times 2 \times \cos 60° + 2 \times 2 \times \cos 90°$$
$$= 2 + 0$$
$$= 2$$