Perth Academy

Mathematics Department

Higher

Key Points

The Circle

The Circle

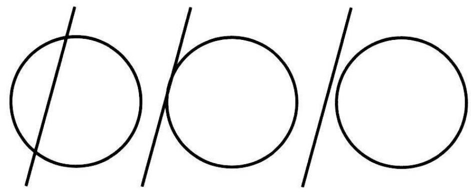
1 The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This is known as the **distance formula**.

2 The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

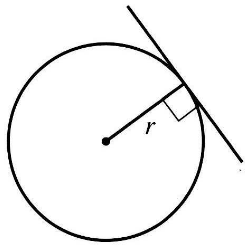
3 The equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

4 $x^2 + y^2 + 2gx + 2fy + c = 0$ (where g, f and c are constants) is the **general equation** of a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$, provided $g^2 + f^2 - c > 0$.

5 A straight line and a circle may have two, one or no points of intersection. Tangency can be determined by solving the linear and circle equations simultaneously and considering the discriminant.
If $b^2 - 4ac = 0$ the line is a tangent.



6 The equation of a tangent at a point on a circle can be found from its gradient and the coordinates of the point of contact using the relationship $m_{rad} \times m_{tan} = -1$.



Example 1

Find the equation of a circle with centre C(5, −2) and passing through the point D(1, −5).

Solution

Length of radius $= CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= CD = \sqrt{(1 - 5)^2 + (-5 - (-2))^2}$
$= CD = \sqrt{16 + 9}$
$= CD = 5$

Equation of circle is $(x - 5)^2 + (y - (-2))^2 = 5^2$
$(x - 5)^2 + (y + 2)^2 = 25$

Example 2

Find the radius and the centre of the circle with equation
$x^2 + y^2 + 2x - 6y + 1 = 0$.

Solution

From the general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = 2$ therefore $g = 1$

$2f = -6$ therefore $f = -3$

$c = 1$

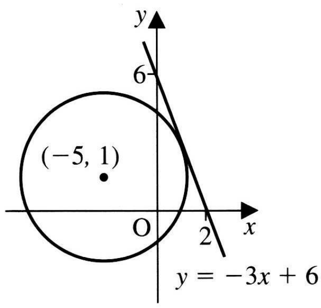
The centre of the circle is $(-g, -f)$, that is $(-1, 3)$

$$
\begin{aligned}
\text{The radius of the circle} &= \sqrt{g^2 + f^2 - c} \\
&= \sqrt{1^2 + (-3)^2 - 1} \\
&= \sqrt{9} \\
&= 3
\end{aligned}
$$

Example 3

(a) Show that the straight line with equation $y = -3x + 6$ is a tangent to the circle with equation $x^2 + y^2 + 10x - 2y - 14 = 0$.

(b) Find the point of contact.



Solution

(a) Substitute $(-3x + 6)$ for y in the circle equation.

$x^2 + (-3x + 6)^2 + 10x - 2(-3x + 6) - 14 = 0$

$x^2 + 9x^2 - 36x + 36 + 10x + 6x - 12 - 14 = 0$

$10x^2 - 20x + 10 = 0$

To examine the roots of this quadratic find the discriminant.

$$
\begin{aligned}
b^2 - 4ac &= (-20)^2 - 4 \times 10 \times 10 \\
&= 400 - 400 \\
&= 0
\end{aligned}
$$

Since the discriminant is zero the roots are equal. There is only one point of intersection. The line is a tangent to the circle.

(b) Solving the equation $10x^2 - 20x + 10 = 0$

$10(x - 1)(x - 1) = 0$

The quadratic has equal roots with $x = 1$

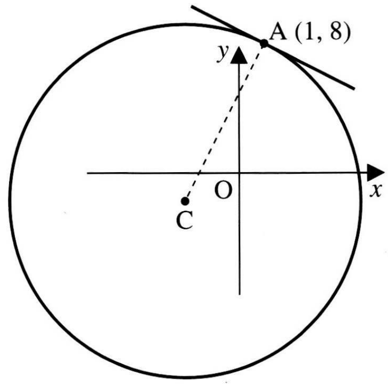
$y = -3 \times 1 + 6$

$y = 3$

The point of contact is $(1, 3)$.

Example 4

The point A(1, 8) lies on the circle with equation $x^2 + y^2 + 8x + 4y - 105 = 0$. Find the equation of the tangent at A.



Solution

The centre of circle $x^2 + y^2 + 8x + 4y - 105 = 0$ is C(−4, −2).

$$m_{CA} = \frac{8 - (-2)}{1 - (-4)} = \frac{10}{5} = 2$$

Since the tangent is perpendicular to the radius,

$m_{tan} \times m_{rad} = -1$

So $m_{tan} = -\frac{1}{2}$

Equation of tangent at A is $y - 8 = -\frac{1}{2}(x - 1)$

$$2y - 16 = -x + 1$$

$$x + 2y - 17 = 0$$