Perth Academy

Mathematics Department

Higher

Key Points

Addition Formulae

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1 $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

2 $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

3 $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

4 $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

5 $\sin 2\alpha = 2\sin\alpha\cos\alpha$

6 $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$= 2\cos^2\alpha - 1$
$= 1 - 2\sin^2\alpha$

7 $\cos^2\alpha = \frac{1}{2}(1 + \cos 2\alpha)$

8 $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$

Example 1

Solve algebraically $2\sin 2x° + 1 = 0$ where $0 \leqslant x \leqslant 360$.

Solution

$2\sin 2x° + 1 = 0$

$2\sin 2x° = -1$

$\sin 2x° = -\frac{1}{2}$

Since $\sin 2x°$ is negative the solutions are in the third and fourth quadrants.

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Also, since $0 \leqslant x \leqslant 360$ then $0 \leqslant 2x \leqslant 720$

$2x = 210$ or 330 or $(210 + 360)$ or $(330 + 360)$

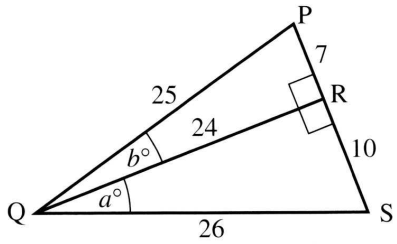
$x = 105$ or 165 or 295 or 345

Example 2

The diagram shows the cross-section of an adjustable ramp which is made from two right-angled triangles, PQR and RQS. Angle RQS = $a°$ and PQR = $b°$.

(a) Find the exact value of $\sin(a+b)°$.

(b) Hence calculate the height of the ramp.



Solution

(a) $\sin(a+b)° = \sin a° \cos b° + \cos a° \sin b°$

$$= \frac{10}{26} \times \frac{24}{25} + \frac{24}{26} \times \frac{7}{25}$$

$$= \frac{240 + 168}{650}$$

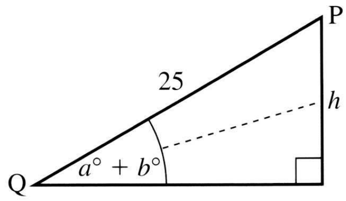
$$= \frac{408}{650}$$

(b) $\sin(a+b)° = \dfrac{h}{25}$

$$h = 25 \sin(a+b)°$$

$$h = 25 \times \frac{408}{650}$$

$$h = 15.69 \text{ m}$$



Example 3

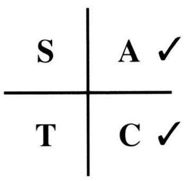
(a) Express $\cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6}$ in the form $\cos(A+B)$.

(b) Hence solve the equation $\cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6} = \frac{1}{2}$ for $0 \leqslant x \leqslant 2\pi$.

Solution

(a) $\cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6} = \cos\left(x + \frac{\pi}{6}\right)$

(b) $\cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6} = \frac{1}{2}$



$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{6} = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

$$x = \frac{\pi}{3} - \frac{\pi}{6} \text{ or } \frac{5\pi}{3} - \frac{\pi}{6}$$

$$x = \frac{\pi}{6} \text{ or } \frac{3\pi}{2}$$

Example 4

Find the exact value of cos15°.

Solution

$$\cos 15° = \cos(45 - 30)°$$
$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

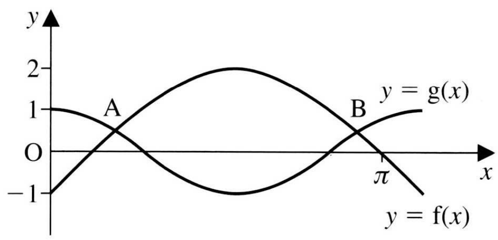
Example 5

The diagram shows the graphs of $f(x) = 3 \sin x - 1$ and $g(x) = \cos 2x$ for $0 \leqslant x \leqslant \pi$.

(a) Solve algebraically the equation $3 \sin x - 1 = \cos 2x$.

(b) Hence find the coordinates of A and B.

(c) For what values of x in the interval $0 \leqslant x \leqslant \pi$ is $\cos 2x \leqslant 3 \sin x - 1$?



Solution

(a)
$$3 \sin x - 1 = \cos 2x$$
$$3 \sin x - 1 = 1 - 2 \sin^2 x$$
$$2 \sin^2 x + 3 \sin x - 2 = 0$$
$$(2 \sin x - 1)(\sin x + 2) = 0$$
$$\sin x = \frac{1}{2} \text{ or } \sin x = -2$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6} \text{ (as } \sin x = -2 \text{ has no solution)}$$

(b) When $x = \frac{\pi}{6}$, $g(x) = \cos \frac{\pi}{3} = \frac{1}{2}$, so A is $\left(\frac{\pi}{6}, \frac{1}{2}\right)$

When $x = \frac{5\pi}{6}$, $g(x) = \cos \frac{5\pi}{3} = \frac{1}{2}$, so B is $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$

(c) For the interval $0 \leqslant x \leqslant \pi$, $g(x)$ lies below $f(x)$ when

$$\frac{\pi}{6} \leqslant x \leqslant \frac{5\pi}{6}$$

Example 6

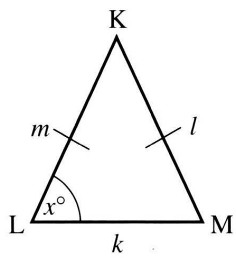
Solve $3 \cos 2x° = \cos x° - 1$, for $0 \leqslant x \leqslant 360$. [Higher]

Solution

$$3 \cos 2x° = \cos x° - 1$$
$$3(2 \cos^2 x° - 1) = \cos x° - 1$$
$$6 \cos^2 x° - 3 - \cos x° + 1 = 0$$
$$6 \cos^2 x° - \cos x° - 2 = 0$$
$$(2 \cos x° + 1)(3 \cos x° - 2) = 0$$
$$2 \cos x° + 1 = 0 \text{ or } 3 \cos x° - 2 = 0$$
$$\cos x° = -\tfrac{1}{2} \text{ or } \cos x° = \tfrac{2}{3}$$
$$x = 120, 240 \text{ or } x = 48.2, 311.8$$

Example 7

The diagram shows an isosceles triangle KLM in which KL = KM and angle KLM = $x°$.



(a) Show that $\sin \dfrac{x°}{m} = \dfrac{\sin 2x°}{k}$

(b) (i) State the value of x when $k = m$.
(ii) Using the fact that $k = m$, solve the equation in (a) above to justify your stated value of x.

Solution

(a) Since the triangle is isosceles, angle KML = $x°$, angle LKM = $(180 - 2x)°$

Using the sine rule: $$\frac{\sin L}{l} = \frac{\sin K}{k}$$
$$\frac{\sin x°}{m} = \frac{\sin(180 - 2x)°}{k}$$

Since $\sin(180 - 2x)° = \sin 2x°$ $$\frac{\sin x°}{m} = \frac{\sin 2x°}{k}$$

(b) (i) When $k = m$, triangle KLM is equilateral, so $x = 60$

(ii) $\dfrac{\sin x°}{m} = \dfrac{\sin 2x°}{k}$ becomes $\dfrac{\sin x°}{m} = \dfrac{\sin 2x°}{m}$

Hence $\sin x° = \sin 2x°$
$$\sin x° = 2 \sin x° \cos x°$$
$$\sin x° - 2 \sin x° \cos x° = 0$$
$$\sin x°(1 - 2 \cos x°) = 0$$
$$\sin x° = 0 \text{ or } \cos x° = \tfrac{1}{2}$$
$$x = 0 \quad \text{or} \quad x = 60$$

Clearly $x = 60$ since the required angle is acute.