Perth Academy

Mathematics Department

Higher

Key Points

Integration

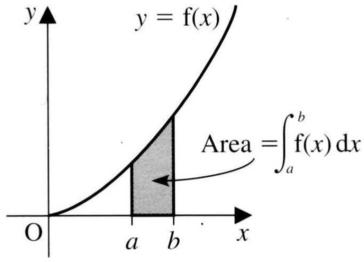
Integration

1 $\int ax^n \, dx = a \int x^n \, dx = \dfrac{ax^{n+1}}{(n+1)} + C \quad (n \neq -1)$, where a is a constant.

2 $\int (f(x) + g(x)) \, dx = \int (x) \, dx + \int g(x) \, dx$

3 The notation for the area between the graph of $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is $\int_a^b f(x) \, dx$.

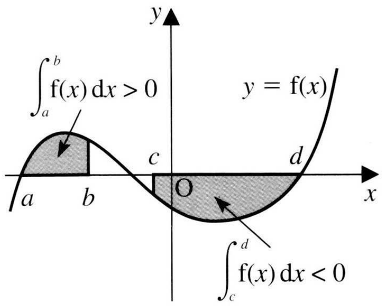
This is called a **definite integral**.
a and b are the **lower** and **upper limits of integration**, respectively.



4 The Fundamental Theorem of Calculus.
If F(x) is the anti-derivative of f(x) then

$$\int_a^b f(x) \, dx = F(b) - F(a) \quad (a \leqslant x \leqslant b)$$

5 When calculated by integration:
- areas above the x-axis are positive
- areas below the x-axis are negative.

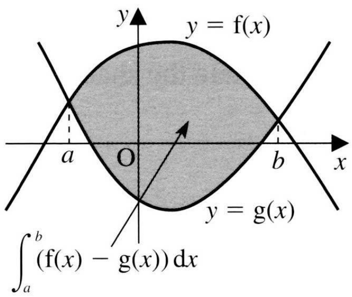


6 When calculating the area between a curve and the x-axis:
- make a sketch
- calculate areas above and below the x-axis **separately**
- ignore negative signs and add.

7 The area enclosed between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b (f(x) - g(x)) \, dx$$

when $f(x) \geqslant g(x)$ and $a \leqslant x \leqslant b$.



Example 1

Integrate $\int \sqrt{x} - \frac{2}{x^3}\,dx$

Solution

$\int \sqrt{x} - \frac{2}{x^3}\,dx$

$= \int x^{\frac{1}{2}} - 2x^{-3}\,dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{-2}}{-2} + C$

$= \frac{2x^{\frac{3}{2}}}{3} + \frac{1}{x^2} + C$

Example 2

Integrate $\int \frac{4x - x^{\frac{3}{2}}}{2\sqrt{x}}\,dx$

$\int \frac{4x - x^{\frac{3}{2}}}{2\sqrt{x}}\,dx$

$= \int \frac{4x}{2x^{\frac{1}{2}}} - \frac{x^{\frac{3}{2}}}{2x^{\frac{1}{2}}}\,dx$

$= \int 2x^{\frac{1}{2}} - \frac{x}{2}\,dx$

$= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{4} + C$

$= \frac{4x^{\frac{3}{2}}}{3} - \frac{x^2}{4} + C$

Example 3

Evaluate $\int_1^2 (3x - 1)(x + 5)\,dx$

Solution

$\int_1^2 (3x - 1)(x + 5)\,dx$

$= \int_1^2 3x^2 + 14x - 5\,dx$

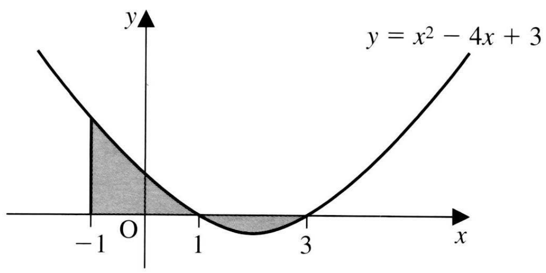
$= [x^3 + 7x^2 - 5x]_1^2$

$= (8 + 28 - 10) - (1 + 7 - 5)$

$= 23$

Example 4

Calculate the shaded area in the diagram.



Solution

Area above x-axis:

$\int_{-1}^{1} x^2 - 4x + 3\,dx$

$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{-1}^{1}$

$= \left(\frac{1}{3} - 2 + 3\right) - \left(\frac{-1}{3} - 2 - 3\right)$

$= 6\frac{2}{3}$

Total area $= 6\frac{2}{3} + 1\frac{1}{3} = 8$ units^2

Area below x-axis:

$\int_{1}^{3} x^2 - 4x + 3\,dx$

$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{1}^{3}$

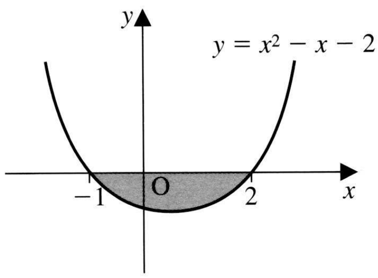
$= (9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right)$

$= -1\frac{1}{3}$

Note: Ignore negative sign for area below the x-axis.

Example 5

Find the area enclosed by $y = x^2 - x - 2$ and the x-axis.



Solution

The graph cuts the x-axis when $x^2 - x - 2 = 0$

$$(x + 1)(x - 2) = 0$$

so $\quad x = -1 \quad$ or $\quad x = 2$

Area: $\displaystyle\int_{-1}^{2} x^2 - x - 2 \, dx$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}$$

$$= \left(\frac{8}{3} - 2 - 4\right) - \left(-\frac{1}{3} - \frac{1}{2} + 2\right)$$

$$= -4\frac{1}{2}$$

So the area is $4\frac{1}{2}$ units^2

Example 6

Find the area enclosed by the graphs of $y = x + 1$ and $y = 5 - 2x - x^2$.

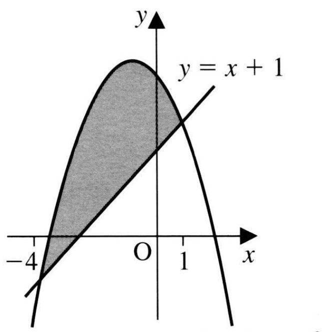
Solution

The graphs intersect where $x + 1 = 5 - 2x - x^2$

$$x + 3x - 4 = 0$$

$$(x + 4)(x - 1) = 0$$

so $\quad x = -4 \quad$ or $\quad x = 1$



Area: $\displaystyle\int_{-4}^{1} (5 - 2x - x^2) - (x + 1) \, dx$

$$= \int_{-4}^{1} 4 - 3x - x^2 \, dx$$

$$= \left[4x - \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-4}^{1}$$

$$= \left(4 - \frac{3}{2} - \frac{1}{3}\right) - \left(-16 - 24 + \frac{64}{3}\right)$$

$$= 20\frac{5}{6} \text{ units}^2$$

Example 7

Determine p given that $\int_1^p \sqrt{x}\,dx = 42.$ [Higher]

Solution

$$\int_1^p \sqrt{x}\,dx = 42$$

$$\int_1^p x^{\frac{1}{2}}\,dx = 42$$

$$\left[\frac{2x^{\frac{3}{2}}}{3}\right]_1^p = 42$$

$$\left(\frac{2p^{\frac{3}{2}}}{3}\right) - \left(\frac{2}{3}\right) = 42$$

$$\frac{2p^{\frac{3}{2}}}{3} = \frac{128}{3}$$

$$p^{\frac{3}{2}} = 64$$

$$p = 16$$

Example 8

The gradient of a tangent to a curve is given by $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

If the curve passes through the point (4, 3) find its equation. [Higher]

Solution

$$y = \int \frac{1}{2\sqrt{x}}\,dx = \int \frac{x^{-\frac{1}{2}}}{2}\,dx$$

$$= x^{\frac{1}{2}} + C$$

$$= \sqrt{x} + C$$

Substituting (4, 3) in $y = \sqrt{x} + C$

$$3 = \sqrt{4} + C$$

$$C = 1$$

The equation of the curve is $y = \sqrt{x} + 1$