Perth Academy

Mathematics Department

Higher

Key Points

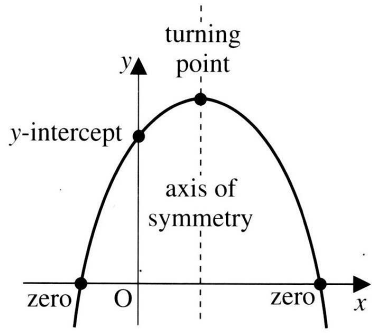
Quadratic Functions

Quadratic Functions

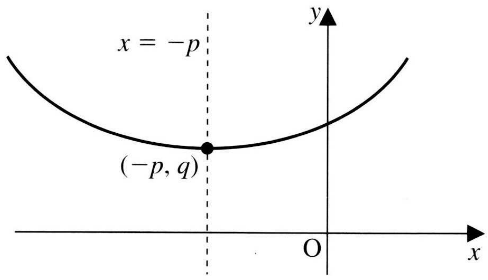
1 The graph of a quadratic function $y = ax^2 + bx + c$ is a **parabola**.
If $a > 0$ the parabola is $\cup$ shaped and the turning point is a minimum.
If $a < 0$ the parabola is $\cap$ shaped and the turning point is a maximum.

2 To sketch and anotate a parabola $y = ax^2 + bx + c$ we need to identify where possible:

- whether the shape is $\cup$ $(a > 0)$ or $\cap$ $(a < 0)$
- the coordinates of the y-intercept, $(0, c)$
- the zeros of the function by solving $ax^2 + bx + c = 0$
- the equation of the axis of symmetry
- the coordinates of the turning point.



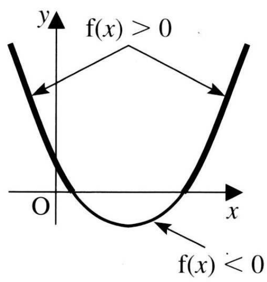
3 When the equation $y = ax^2 + bx + c$ is written in the form $y = a(x + p)^2 + q$, the axis of symmetry is $x = -p$ and the turning point is at $(-p, q)$.

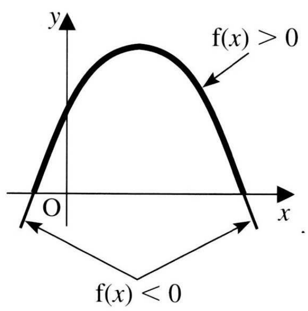


4 Quadratic equations may be solved by

- using the graph
- factorising
- completing the square
- using the quadratic formula.

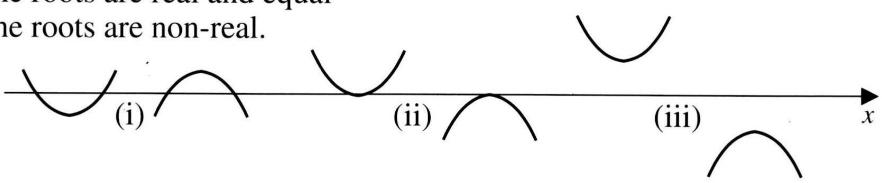
5 A quadratic inequation can be solved using a sketch of the quadratic function.





6 If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a \neq 0$

7 For the quadratic equation $ax^2 + bx + c = 0$, $b^2 - 4ac$ is called the **discriminant**:
(i) if $b^2 - 4ac > 0$, the roots are real and unequal
(ii) if $b^2 - 4ac = 0$, the roots are real and equal
(iii) if $b^2 - 4ac < 0$, the roots are non-real.



Example 1

Determine the nature of the roots of each of these equations, using the discriminant.

(a) $x^2 - 6x + 8 = 0$ (b) $x^2 - 6x + 9 = 0$ (c) $x^2 - 6x + 10 = 0$

Solution

(a) $x^2 - 6x + 8 = 0 \quad a = 1, b = -6, c = 8$

$b^2 - 4ac = (-6)^2 - (4 \times 1 \times 8) = 36 - 32 = 4$

discriminant is positive, so there are two unequal (distinct) real roots.

(b) $x^2 - 6x + 9 = 0 \quad a = 1, b = -6, c = 9$

$b^2 - 4ac = (-6)^2 - (4 \times 1 \times 9) = 36 - 36 = 0$

discriminant is zero, so roots are real and equal.

(c) $x^2 - 6x + 10 = 0 \quad a = 1, b = -6, c = 10$

$b^2 - 4ac = (-6)^2 - (4 \times 1 \times 10) = 36 - 40 = -4$

discriminant is negative, so roots are non–real.

Example 2

For what values of p does the equation $x^2 - 2px + (2 - p) = 0$ have non–real roots?

Solution

$x^2 - 2px + (2 - p) = 0$

$a = 1, b = -2p, c = 2 - p$

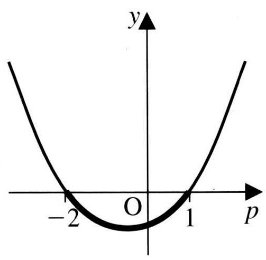
For non–real roots $b^2 - 4ac < 0$, so

$(-2p)^2 - 4(2 - p) < 0$

$4p^2 - 8 + 4p < 0$

$4(p^2 + p - 2) < 0$

$4(p - 1)(p + 2) < 0$



From the graph of $4(p - 1)(p + 2) < 0$ when $-2 < p < 1$

The equation has non–real roots when $-2 < p < 1$

Example 3

Find values of q so that $\dfrac{1}{x^2 - x + 1} = q$ has two equal roots.

Solution

Cross-multiplying gives $q(x^2 - x + 1) = 1$

$qx^2 - qx + q = 1$

$qx^2 - qx + (q - 1) = 0$

$a = q, b = -q, c = (q - 1)$

For equal roots $b^2 - 4ac = 0$, so

$(-q)^2 - 4q(q - 1) = 0$

$q^2 - 4q^2 + 4q = 0$

$4q - 3q^2 = 0$

$q(4 - 3q) = 0$

The equation has two equal roots when $q = 0$ or $q = \frac{4}{3}$.

Example 4

Given that k is a real number, show that the roots of the equation $kx^2 + 5x + 5 = k$ are always real numbers.

Solution

$kx^2 + 5x + 5 = k$

$kx^2 + 5x + (5 - k) = 0$

$a = k, b = 5, c = 5 - k$

$$b^2 - 4ac = 25 - 4k(5 - k)$$
$$= 25 - 20k + 4k^2$$
$$= (5 - 2k)^2$$

Since $(5 - 2k)^2$ is a square it has a minimum value zero, therefore $b^2 - 4ac > 0$ and so the roots of the equation are always real.

Example 5

Find the equation of the parabola that passes through $(-1, 0)$, $(5, 0)$ and $(0, -10)$ in the form $y = ax^2 + bx + c$.

Solution

The parabola cuts the x-axis at $x = -1$ and $x = 5$,

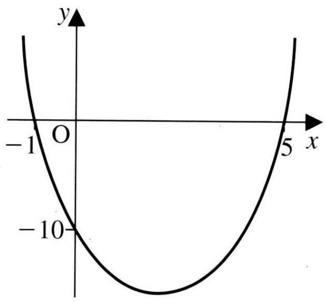
so $k(x + 1)(x - 5) = 0$, where k is a constant

$$y = k(x + 1)(x - 5)$$

When $x = 0, y = -10$

$$k(0 + 1)(0 - 5) = -10$$
$$-5k = -10$$
$$k = 2$$

Hence $y = 2(x + 1)(x - 5) = 2x^2 - 8x - 10$



Example 6

Show that $y = 15 - 7x$ is a tangent to the parabola $y = -x^2 - x + 6$ and find the point of contact.

Solution

The line and parabola meet where $15 - 7x = -x^2 - x + 6$

$$x^2 - 6x + 9 = 0$$

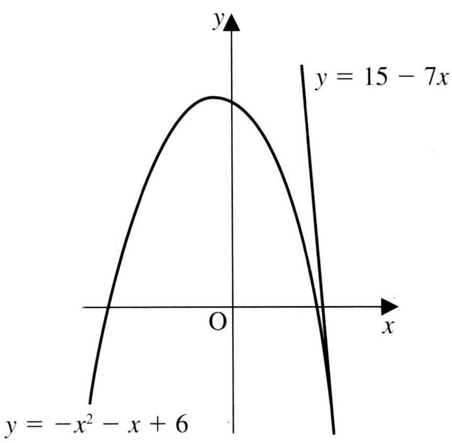
$b^2 - 4ac = (-6)^2 - (4 \times 1 \times 9) = 0$

so there is one point of intersection. Hence the line is a tangent to the parabola.

The line and parabola meet where $x^2 - 6x + 9 = 0$

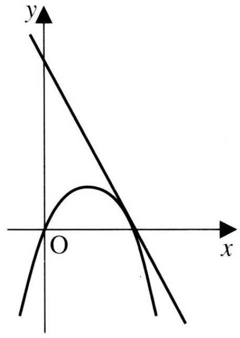
$$(x - 3)(x - 3) = 0$$
$$\text{so } x = 3$$

The point of contact is $(3, -6)$.



Example 7

The line $y = -2x + k$ is a tangent to the parabola $y = 4x - x^2$.
Find the value of k.



Solution

$y = -2x + k$ meets $y = 4x - x^2$ where $-2x + k = 4x - x^2$

$$x^2 - 6x + k = 0$$

Tangency implies equal roots therefore $b^2 - 4ac = 0$

$$(-6) - 4k = 0$$

$$k = 9$$

The equation of the tangent is $y = 9 - 2x$.