Perth Academy
Mathematics Department
Higher
Key Points
Quadratic Functions
Quadratic Functions

1. The graph of a quadratic function $y = ax^2 + bx + c$ is a **parabola**.
   If $a > 0$ the parabola is $\cup$ shaped and the turning point is a minimum.
   If $a < 0$ the parabola is $\cap$ shaped and the turning point is a maximum.

2. To sketch and annotate a parabola $y = ax^2 + bx + c$ we need to identify where possible:
   - whether the shape is $\cup$ ($a > 0$) or $\cap$ ($a < 0$)
   - the coordinates of the $y$-intercept, $(0, c)$
   - the zeros of the function by solving $ax^2 + bx + c = 0$
   - the equation of the axis of symmetry
   - the coordinates of the turning point.

3. When the equation $y = ax^2 + bx + c$ is written in the form $y = a(x + p)^2 + q$, the axis of symmetry is $x = -p$ and the turning point is at $(-p, q)$.

4. Quadratic equations may be solved by
   - using the graph
   - factorising
   - completing the square
   - using the quadratic formula.

5. A quadratic inequation can be solved using a sketch of the quadratic function.

6. If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $a \neq 0$.

7. For the quadratic equation $ax^2 + bx + c = 0$, $b^2 - 4ac$ is called the **discriminant**:
   (i) if $b^2 - 4ac > 0$, the roots are real and unequal
   (ii) if $b^2 - 4ac = 0$, the roots are real and equal
   (iii) if $b^2 - 4ac < 0$, the roots are non-real.
Example 1

Determine the nature of the roots of each of these equations, using the discriminant.
(a) \( x^2 - 6x + 8 = 0 \)  \( a = 1, b = -6, c = 8 \)
\[ b^2 - 4ac = (-6)^2 - (4 \times 1 \times 8) = 36 - 32 = 4 \]
Discriminant is positive, so there are two unequal (distinct) real roots.

(b) \( x^2 - 6x + 9 = 0 \)  \( a = 1, b = -6, c = 9 \)
\[ b^2 - 4ac = (-6)^2 - (4 \times 1 \times 9) = 36 - 36 = 0 \]
Discriminant is zero, so roots are real and equal.

(c) \( x^2 - 6x + 10 = 0 \)  \( a = 1, b = -6, c = 10 \)
\[ b^2 - 4ac = (-6)^2 - (4 \times 1 \times 10) = 36 - 40 = -4 \]
Discriminant is negative, so roots are non-real.

Example 2

For what values of \( p \) does the equation \( x^2 - 2px + (2 - p) = 0 \) have non-real roots?

Solution
\[ x^2 - 2px + (2 - p) = 0 \]
\( a = 1, b = -2p, c = 2 - p \)
For non-real roots \( b^2 - 4ac < 0 \), so
\[ (-2p)^2 - 4(2 - p) < 0 \]
\[ 4p^2 - 8 + 4p < 0 \]
\[ 4(p^2 + p - 2) < 0 \]
\[ 4(p - 1)(p + 2) < 0 \]
From the graph of \( 4(p - 1)(p + 2) < 0 \) when \(-2 < p < 1\)
The equation has non-real roots when \(-2 < p < 1\)

Example 3

Find values of \( q \) so that \( \frac{1}{x^2 - x + 1} = q \) has two equal roots.

Solution
Cross-multiplying gives \( q(x^2 - x + 1) = 1 \)
\[ qx^2 - qx + q = 1 \]
\[ qx^2 - qx + (q - 1) = 0 \]
\( a = q, b = -q, c = (q - 1) \)
For equal roots \( b^2 - 4ac = 0 \), so
\[ (-q)^2 - 4q(q - 1) = 0 \]
\[ q^2 - 4q^2 + 4q = 0 \]
\[ 4q - 3q^2 = 0 \]
\[ q(4 - 3q) = 0 \]
The equation has two equal roots when \( q = 0 \) or \( q = \frac{4}{3} \).
Example 4
Given that $k$ is a real number, show that the roots of the equation $kx^2 + 5x + 5 = k$ are always real numbers.

Solution
$$kx^2 + 5x + 5 = k$$
$$kx^2 + 5x + (5 - k) = 0$$
$$a = k, b = 5, c = 5 - k$$
$$b^2 - 4ac = 25 - 4k(5 - k)$$
$$= 25 - 20k + 4k^2$$
$$= (5 - 2k)^2$$
Since $(5 - 2k)^2$ is a square it has a minimum value zero, therefore $b^2 - 4ac > 0$ and so the roots of the equation are always real.

Example 5
Find the equation of the parabola that passes through $(-1, 0)$, $(5, 0)$ and $(0, -10)$ in the form $y = ax^2 + bx + c$.

Solution
The parabola cuts the $x$-axis at $x = -1$ and $x = 5$, so $k(x + 1)(x - 5) = 0$, where $k$ is a constant
$$y = k(x + 1)(x - 5)$$
When $x = 0, y = -10$
$$k(0 + 1)(0 - 5) = -10$$
$$-5k = -10$$
$$k = 2$$
Hence $y = 2(x + 1)(x - 5) = 2x^2 - 8x - 10$

Example 6
Show that $y = 15 - 7x$ is a tangent to the parabola $y = -x^2 - x + 6$ and find the point of contact.

Solution
The line and parabola meet where $15 - 7x = -x^2 - x + 6$
$$x^2 - 6x + 9 = 0$$
$$b^2 - 4ac = (-6)^2 - (4 \times 1 \times 9) = 0$$
so there is one point of intersection. Hence the line is a tangent to the parabola.
The line and parabola meet where $x^2 - 6x + 9 = 0$
$$(x - 3)(x - 3) = 0$$
so $x = 3$
The point of contact is $(3, -6)$. 
Example 7
The line $y = -2x + k$ is a tangent to the parabola $y = 4x - x^2$.
Find the value of $k$.

Solution
$y = -2x + k$ meets $y = 4x - x^2$ where $-2x + k = 4x - x^2$
$x^2 - 6x + k = 0$

Tangency implies equal roots therefore $b^2 - 4ac = 0$
$(-6)^2 - 4k = 0$
$k = 9$

The equation of the tangent is $y = 9 - 2x$. 