Perth Academy

Mathematics Department

Higher

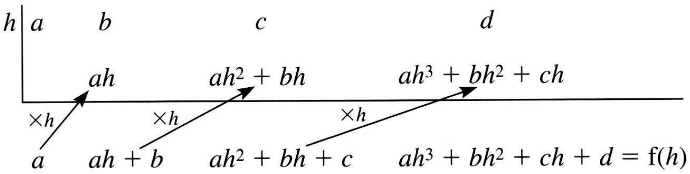
Key Points

Polynomials

Polynomials

1 A **root** of a polynomial function, $f(x)$, is a value of x for which $f(x) = 0$.

2 When $ax^3 + bx^2 + cx + d$ is divided by $x - h$, the quotient and remainder can be found by synthetic division:

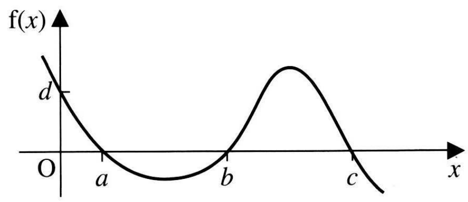


The quotient is $ax^2 + (ah + b)x + (ah^2 + bh + c)$ and the remainder is $ah^3 + bh^2 + ch + d$.

3 The remainder theorem: If a polynomial $f(x)$ is divided by $(x - h)$ the remainder is $f(h)$.

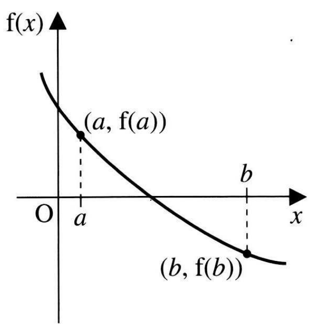
4 The factor theorem:
If $f(h) = 0$ then $x - h$ is a factor of $f(x)$.
Conversely, if $(x - h)$ is a factor of $f(x)$ then $f(h) = 0$.

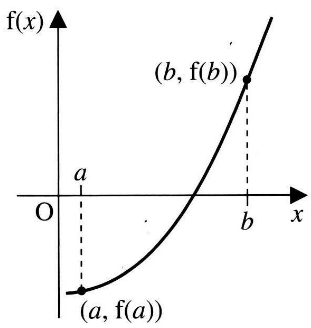
5 The equation of a polynomial may be established from its graph:



$f(x) = k(x - a)(x - b)(x - c)$ is the general equation for the family of curves. k can be found by substituting $(0, d)$ in $f(x)$.

6 A root of a polynomial lies between $x = a$ and $x = b$ if $f(a) > 0$ and $f(b) < 0$ or if $f(a) < 0$ and $f(b) > 0$.





Example 1
Show that $(x - 5)$ is a factor of $f(x) = 2x^3 + x^2 - 50x - 25$ and express $f(x)$ in fully factorised form.

Solution

5	2	1	−50	−25	
		×5 → 10	×5 → 55	×5 → 25	
	2	11	5	**0**	so $f(5) = 0$

Since $f(5) = 0$, $(x - 5)$ is a factor of $2x^3 + x^2 - 50x - 25$

Hence $f(x) = (x - 5)(2x^2 + 11x + 5)$

$= (x - 5)(2x + 1)(x + 5)$

Example 2
Express $(8x^4 + 2x^2 - 4x - 3) \div (2x - 1)$ in the form $(2x - 1)Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.

Solution
$2x - 1 = 0$ has root $x = \frac{1}{2}$

Hence

$\frac{1}{2}$	8	0	2	−4	−3	coefficient of x^3 is zero
		4	2	2	−1	
	8	4	4	−2	−4	$= (f\frac{1}{2})$

Hence $f(x) = (x - \frac{1}{2})(8x^3 + 4x^2 + 4x - 2) - 4$

$= (2x - 1)(4x^3 + 2x^2 + 2x - 1) - 4$

Example 3
Find the roots of $x^4 + x^3 - 9x^2 - 7x + 14 = 0$.

Solution
To find the roots consider factors of 14, i.e. $\pm 1, \pm 2, \pm 7$

1	1	1	−9	−7	14
		1	2	−7	−14
	1	2	−7	−14	**0**

$f(x) = (x - 1)(x^3 + 2x^2 - 7x - 14)$

To find the factors of $(x^3 + 2x^2 - 7x - 14)$:

−2	1	2	−7	−14
		−2	0	14
	1	0	−7	**0**

$f(x) = (x - 1)(x + 2)(x^2 - 7)$

Hence, $x - 1 = 0$ or $x + 2 = 0$ or $x^2 - 7 = 0$

$x = 1$ $\quad x = -2$ $\quad x = \pm\sqrt{7}$

The roots are $-\sqrt{7}, -2, 1, \sqrt{7}$.

Example 4

Given that $(x + 1)$ and $(x - 3)$ are factors of $f(x) = 2x^3 - 5x^2 + px + q$, find p and q.

Solution

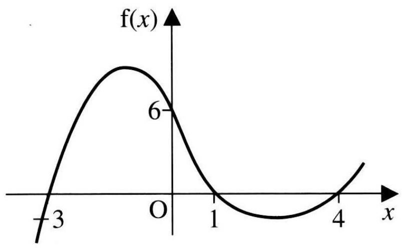
-1	2	-5	p	q
		-2	7	$-p - 7$
	2	-7	$p + 7$	$q - p - 7 = 0$

3	2	-5	p	q
		6	3	$3p + 9$
	2	1	$p + 3$	$q + 3p + 9 = 0$

Solving simultaneous equations $q - p = 7$ and $q + 3p = -9$ gives $p = -4$ and $q = 3$.

Example 5

From the graph find an expression for $f(x)$.



Solution

The graph has zeros at $x = -3$, $x = 1$ and $x = 4$ hence
$f(x) = k(x + 3)(x - 1)(x - 4)$

From the graph, $f(0) = 6$

so $k(0 + 3)(0 - 1)(0 - 4) = 6$

$$12k = 6$$

$$k = \tfrac{1}{2}$$

Hence $f(x) = \frac{1}{2}(x + 3)(x - 1)(x - 4)$

$$= \tfrac{1}{2}x^3 - x^2 + \frac{11x}{2} + 6$$

Example 6

Sketch the graph of $y = 3x^3 - 9x + 6$.

Solution

y-intercept

When $x = 0$, $y = 6$ so the y-intercept is at $(0, 6)$.

x-intercept

Solve $3x^3 - 9x + 6 = 0$

$$3(x^3 - 3x + 2) = 0$$

1	1	0	−3	2
		1	1	−2
	1	1	−2	**0**

$$3(x - 1)(x^2 + x - 2) = 0$$
$$3(x - 1)(x + 2)(x - 1) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 1 \quad \text{or} \quad x = -2$$

The graph cuts the x-axis at $(1, 0)$ and $(-2, 0)$.

Stationary points

$$y = 3x^3 - 9x + 6$$
$$\frac{dy}{dx} = 9x^2 - 9$$

For stationary points $\frac{dy}{dx} = 0$

$$9x^2 - 9 = 0$$
$$9(x + 1)(x - 1) = 0$$
$$x = -1 \quad \text{or} \quad x = 1$$

When $x = -1$, $y = 12$

When $x = 1$, $y = 0$

x	-1^-	-1	-1^+
$\frac{dy}{dx}$	+	0	−
slope	/	—	\

maximum turning point $(-1, 12)$

x	1^-	1	1^+
$\frac{dy}{dx}$	−	0	+
slope	\	—	/

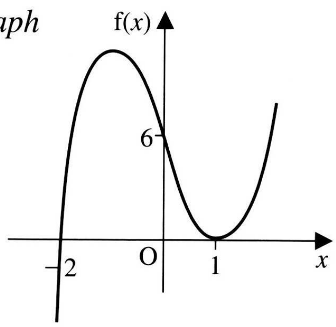
minimum turning point $(1, 0)$

Large positive and negative x

As $x \to -\infty$, $3x^3 - 9x + 6 \to x^3 \to -\infty$

As $x \to \infty$, $3x^3 - 9x + 6 \to x^3 \to \infty$

Graph



Example 7

For the function $f(x) = x^3 - x^2 + 2x + 1$ show there is a real root between 0 and -1. Find this root to two decimal places.

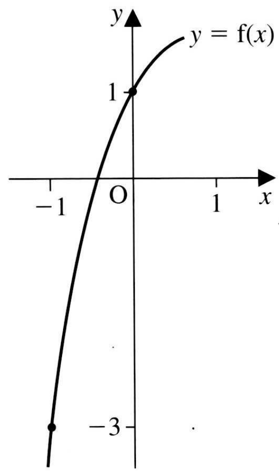
Solution

$f(0) = 1$, which is positive so the graph is above the x-axis.

$f(-1) = -3$, which is negative so the graph is below the x-axis.

Hence the graph crosses the x-axis between 0 and -1.

Evaluate $f(x)$ for values between 0 and -1.



x	$f(x)$	Root lies between
0	1	
−1	−3	0 and −1
−0.3	0.283	−1 and −0.3
−0.4	−0.024	−0.3 and −0.4
−0.38	0.0407	−0.38 and −0.4
−0.39	0.00858	−0.39 and −0.4
−0.395	−0.00765	−0.39 and −0.395

The root is −0.39 to two decimal places.