



Perth Academy

Mathematics Department

Higher

Key Points

Recurrence Relations

Recurrence Relations

- 1** A recurrence relation describes a sequence in which each term is a function of previous terms.
- 2** A sequence defined by $u_{n+1} = au_n + b$, $a \neq 0$ is called a **linear recurrence relation**.
- 3** For a linear recurrence relation $u_{n+1} = au_n + b$:
If $-1 < a < 1$, then u_n tends to a limit
If u_n tends to a limit, L , then $L = \frac{b}{1-a}$

Example 1

A sequence is defined by the recurrence relation

$$u_{n+1} = 0.5u_n + 12, u_0 = 10.$$

- (a) Find the next four terms of this recurrence relation.
- (b) Find the limit of this recurrence relation if it exists.

Solution

$$\begin{aligned} \text{(a) } u_1 &= 0.5u_0 + 12 & u_2 &= 0.5u_1 + 12 \\ &= 0.5 \times 10 + 12 & &= 0.5 \times 17 + 12 \\ &= 17 & &= 20.5 \\ u_3 &= 0.5u_2 + 12 & u_4 &= 0.5u_3 + 12 \\ &= 0.5 \times 20.5 + 12 & &= 0.5 \times 22.25 + 12 \\ &= 22.25 & &= 23.125 \end{aligned}$$

- (b) $a = 0.5$, so $-1 < a < 1$ and therefore a limit exists.

$$\begin{aligned} L &= \frac{b}{1-a} & \text{or} & & L &= 0.5L + 12 \\ &= \frac{12}{1-0.5} & & & L - 0.5L &= 12 \\ &= 24 & & & 0.5L &= 12 \\ & & & & L &= 24 \end{aligned}$$

Example 2

In an old car it is estimated that oil leaks from the engine at a rate of 20% per week. The engine contains 5 litres of oil immediately after being filled. The engine will be irreparably damaged if the volume of oil drops below 3.5 litres. If the engine is topped up with 0.5 litres of oil each week:

- write down a recurrence relation that describes the situation above
- find the limit and explain what it means in the context of the question.

Solution

- (a) Let u_n = the volume of oil in the engine after n weeks.

$$u_{n+1} = 0.8u_n + 0.5, u_0 = 5$$

- (b) $a = 0.8$, so $-1 < a < 1$ and therefore a limit exists.

$$\begin{aligned} L &= \frac{b}{1-a} & \text{or} & & L &= 0.8L + 0.5 \\ &= \frac{0.5}{1-0.8} & & & L - 0.8L &= 0.5 \\ &= 2.5 & & & 0.2L &= 0.5 \\ & & & & L &= 2.5 \end{aligned}$$

Since $2.5 < 3.5$, irreparable damage will be done to the engine if this situation continues.

Example 3

The sum of £2000 is placed in an investment account on January 1st and thereafter £50 is placed in the account on the first day of each month.

- Interest at the rate of 0.25% per month is credited to the account on the last day of each month.
 - This interest is calculated on the amount in the account on the first day of the month.
- How much is in the account on June 30th?
 - On what date does the account first exceed £2500?
 - Find a recurrence relation that describes the amount in the account, explaining your notation carefully.

Solution

- (a) Feb 1 balance = $1.0025 \times 2000 + 50 = 2055$
Mar 1 balance = $1.0025 \times 2055 + 50 = 2110.14$
Apr 1 balance = $1.0025 \times 2110.14 + 50 = 2165.41$
May 1 balance = $1.0025 \times 2165.41 + 50 = 2220.83$
Jun 1 balance = $1.0025 \times 2220.83 + 50 = 2276.38$
Jun 30 balance = $1.0025 \times 2276.38 = 2282.07$
The balance on 30th June is £2282.07
- (b) Aug 1 balance = $1.0025 \times 2332.07 + 50 = 2387.90$
Sep 1 balance = $1.0025 \times 2387.90 + 50 = 2443.87$
Oct 1 balance = $1.0025 \times 2443.87 + 50 = 2499.98$
Nov 1 balance = $1.0025 \times 2499.98 + 50 = 2556.23$
Oct 30 balance = $2556.23 - 50 = 2506.23$
The account first exceeds £2500 on 30th October.
- (c) $u_{n+1} = 1.0025u_n + 50$, $u_0 = 2000$ where u_n is the amount in the account on the first day of the n th month after the initial deposit.