Perth Academy

Mathematics Department

Higher

Key Points

Differentiation

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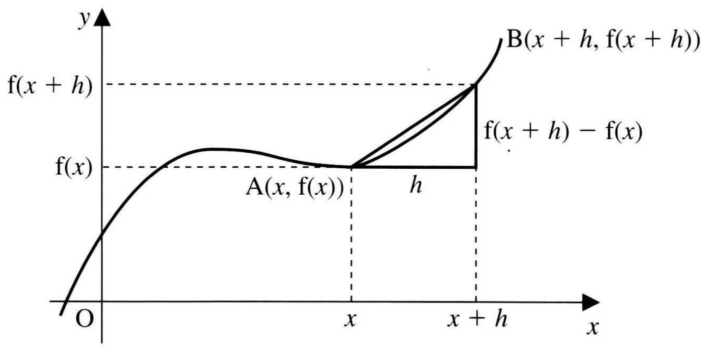
1 The rate of change of a function $f(x)$ can be written as $f'(x)$ and is called the **derived function**.

2 The derivative of a function represents:
- the rate of change of a function
- the gradient of the tangent to the function.

3 $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$



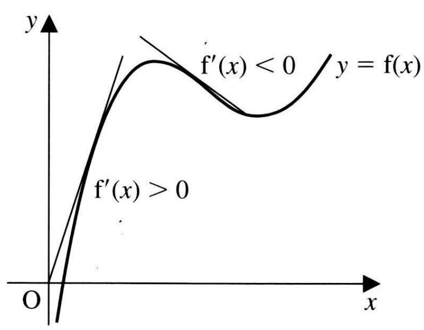
4 If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$, where a is a constant and n is a rational number.

5 If $f(x) = g(x) + h(x)$ then $f'(x) = g'(x) + h'(x)$

6 $\frac{dy}{dx} = f'(x)$

7 Since a tangent is a straight line, its equation may be given by $y - b = m(x - a)$. Hence, to find the equation of the tangent at any point on a curve we need to determine:
- the coordinates (a, b) of the point
- the gradient, m, at that point by finding $\frac{dy}{dx}$

8 For any curve:
If $f'(x) > 0$ in a given interval then $f(x)$ is strictly increasing in that interval.
If $f'(x) < 0$ in a given interval then $f(x)$ is strictly decreasing in that interval.



Example 1

Find $f'(x)$ when $f(x) = 4x^3 + \sqrt{x} + 2$.

Solution

$$f(x) = 4x^3 + \sqrt{x} + 2$$
$$= 4x^3 + x^{\frac{1}{2}} + 2$$
$$f'(x) = 12x^2 + \tfrac{1}{2}x^{-\frac{1}{2}}$$
$$= 12x^2 + \frac{1}{2\sqrt{x}}$$

Example 2

Given that $y = \dfrac{x^5 + 2}{x^2}$, find $\dfrac{dy}{dx}$.

Solution

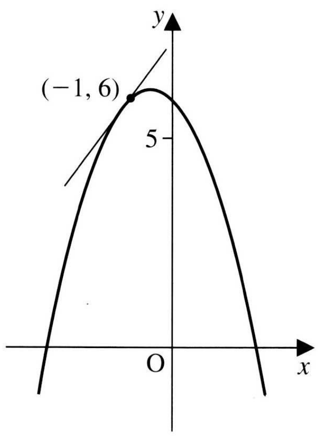
$$y = \frac{x^5 + 2}{x^2}$$
$$= \frac{x^5}{x^2} + \frac{2}{x^2}$$
$$= x^3 + 2x^{-2}$$
$$\frac{dy}{dx} = 3x^2 + (-4)x^{-3}$$
$$= 3x^2 - \frac{4}{x^3}$$

Example 3

The diagram shows the graph of $y = (2 - x)(x + 3)$.

(a) Find the gradient of the tangent at $(-1, 6)$.

(b) Hence find its equation.



Solution

(a) $y = (2 - x)(x + 3) = 6 - x - x^2$

$$\frac{dy}{dx} = -1 - 2x$$

When $x = -1$ $\quad \dfrac{dy}{dx} = -1 - 2 \times (-1)$

$$= 1$$

Hence gradient of tangent $= 1$

(b) $y - b = m(x - a)$

$y - 6 = 1(x - (-1))$

$y = x + 7$

Example 4

Find the rate of change of $f(x) = \dfrac{x^4 - 1}{x}$ at $x = 2$.

Solution

$$f(x) = \frac{x^4 - 1}{x} = \frac{x^4}{x} - \frac{1}{x} = x^3 - x^{-1}$$
$$f'(x) = 3x^2 - (-1)x^{-2}$$
$$= 3x^2 + \frac{1}{x^2}$$
$$f'(2) = 3 \times 2^2 + \frac{1}{2^2}$$
$$= 12\tfrac{1}{4}$$

Example 5

For the curve with equation $y = 2x^3 - 9x^2 - 24x + 6$ find the coordinates of the stationary points and determine their nature.

Solution

$y = 2x^3 - 9x^2 - 24x + 6$

$\frac{dy}{dx} = 6x^2 - 18x - 24$

Stationary points occur when $\frac{dy}{dx} = 0$

$6x^2 - 18x - 24 = 0$
$6(x + 1)(x - 4) = 0$
$x + 1 = 0 \quad x - 4 = 0$
$x = -1 \quad x = 4$

When $x = -1$, $y = 19$ and when $x = 4$, $y = -106$.
The stationary points are $(-1, 19)$ and $(4, -106)$.

Consider the gradient, $\frac{dy}{dx}$, in the neighbourhood of each stationary point.

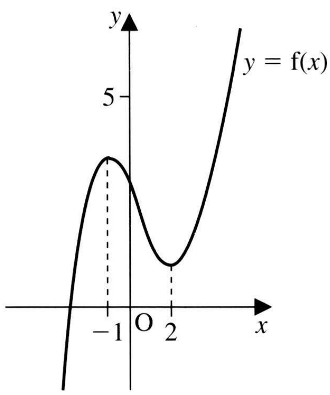
x	-1^-	-1	-1^+
$\frac{dy}{dx}$	+	0	−
slope	/	—	\

x	4^-	4	4^+
$\frac{dy}{dx}$	−	0	+
slope	\	—	/

There is a maximum turning point at $(-1, 19)$ and a minimum turning point at $(4, -106)$.

Example 6

The graph shows $y = f(x)$ for $-2 \leqslant x \leqslant 4$ where $f(x)$ is a cubic function. Sketch the graph of $y = f'(x)$ for the same domain.

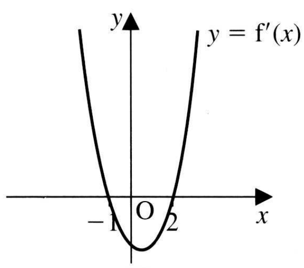


Solution

From the graph of $y = f(x)$ we can determine the following:

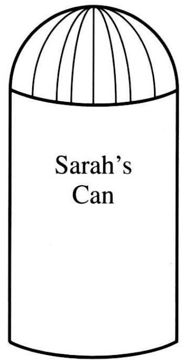
	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$f'(x)$	+ve	zero	−ve	zero	+ve

Hence the graph of $y = f'(x)$ is as shown opposite.



Example 7

Sarah's drinking beaker is in the shape of a cylinder with a hemispherical lid and a circular flat base. The radius of the cylinder is r cm and the height is h cm. The volume of the cylinder is 400 cm^3.



(a) Show that the surface area of plastic, $A(r)$, needed to make the beaker is given by $A(r) = 3\pi r^2 + \frac{800}{r}$.

Note: The curved surface area of a hemisphere of radius r is $2\pi r^2$.

(b) Find the value of r which ensures that the surface area of plastic is minimised. [Higher]

Solution

(a) Area of base $= \pi r^2$

Area of hemisphere $= 2\pi r^2$

Area of side $= 2\pi rh$

To find h:

Volume of cylinder $= \pi r^2 h = 400$

$$h = \frac{400}{\pi r^2}$$

$$\text{Total area} = \pi r^2 + 2\pi r^2 + 2\pi rh$$

$$A(r) = \pi r^2 + 2\pi r^2 + 2\pi r \times \frac{400}{\pi r^2}$$

$$= 3\pi r^2 + \frac{800}{r}$$

(b) For a minimum to occur, $A'(r) = 0$

$$A(r) = 3\pi r^2 + \frac{800}{r}$$

$$= 3\pi r^2 + 800r^{-1}$$

$$A'(r) = 6\pi r - 800r^{-2}$$

$$= 6\pi r - \frac{800}{r^2}$$

$$6\pi r - \frac{800}{r^2} = 0$$

$$6\pi r^3 - 800 = 0$$

$$r^3 = \frac{800}{6\pi}$$

$$r = 3.5 \text{ (to 1 decimal place)}$$

r	3.5^-	3.5	3.5^+
$A'(r)$	−	0	+
slope	\	—	/

Hence a minimum occurs when $r = 3.5$ cm.