14 / 12 / 17

*Unit 2 : Properties of Functions - Lesson 5*

# Asymptotes and Sketching Rational Functions

LI

- Know the various types of asymptotes of rational functions.
- Sketch rational functions with full annotation.

SC

- Long division.
- Graph sketching.

Jan 10-12:11

An asymptote is a line (not necessarily straight) that a function approaches as the x - values approach a certain value

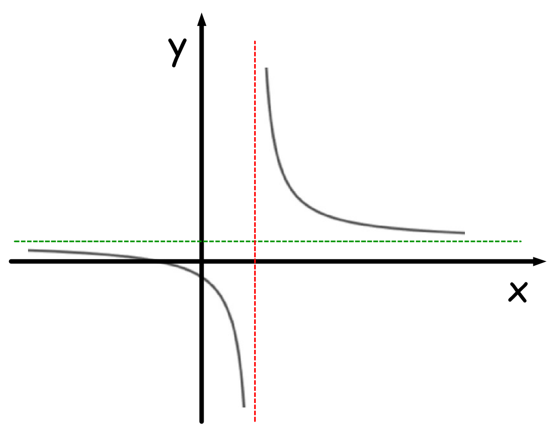
Types

A function $f$ has vertical asymptote $x = A$ if $f$ approaches $\pm\infty$ as $x$ approaches $A$ from values smaller than $A$ $(A^-)$ or $x$ approaches $A$ from values bigger than $A$ $(A^+)$

A vertical asymptote is parallel to the y - axis

A function $f$ has horizontal asymptote $y = B$ if $f$ approaches $B$ as $x$ approaches $\pm\infty$

A horizontal asymptote is parallel to the x - axis




A function $f$ has an oblique asymptote $y = g(x)$ if $f$ approaches $g(x)$ as $x$ approaches $\pm$ infinity

In this course, oblique asymptotes will be straight lines, so the equation will be $y = mx + c$ $(m \neq 0)$

## Rational Functions

Every improper rational function $\dfrac{P(x)}{Q(x)}$ (deg $P \geq$ deg $Q$)

can be written (using long division) in the form :

$$\frac{P(x)}{Q(x)} = W(x) + \frac{R(x)}{Q(x)} \quad (\deg R < \deg Q)$$

Vertical asymptotes : solve $Q(x) = 0$.

$y = W(x)$ will give either a horizontal or oblique asympotote :

Horizontal asymptote : $W(x) = \text{constant}$.

Oblique asymptote : $W(x) = mx + c \ (m \neq 0)$.

## Sketching Rational Functions

When sketching graphs of rational functions, consider :

- Asymptotes.
- Intersections with axes.
- Stationary points.
- Inflexion points.

## Example

Sketch the graph of $y = \dfrac{x^2 - 3x + 3}{x - 2}$.

Long division gives (check !),

$$y = \frac{x^2 - 3x + 3}{x - 2} = (x - 1) + \frac{1}{x - 2}$$

## Asymptotes

VA : $\underline{x = 2}$

As $x \longrightarrow 2^-, y \longrightarrow -\infty$.

As $x \longrightarrow 2^+, y \longrightarrow \infty$.

OA : $\underline{y = x - 1}$

As $x \longrightarrow -\infty, y \longrightarrow (x - 1)^-$.

As $x \longrightarrow \infty, y \longrightarrow (x - 1)^+$.

## Intersections with Axes

$x = 0 \Rightarrow y = -3/2$ : $\underline{(0, -3/2)}$

$y = 0 \Rightarrow x^2 - 3x + 3 = 0$; discriminant $< 0$, so no solutions for $y$.

## Stationary Points

$$y = x - 1 + (x - 2)^{-1}$$

$$\therefore \quad y' = 1 - (x - 2)^{-2}$$

$$y'' = 2(x - 2)^{-3}$$

Solving $y' = 0$ gives two stationary points : $(1, -1)$, $(3, 3)$; the second derivative classifies them.

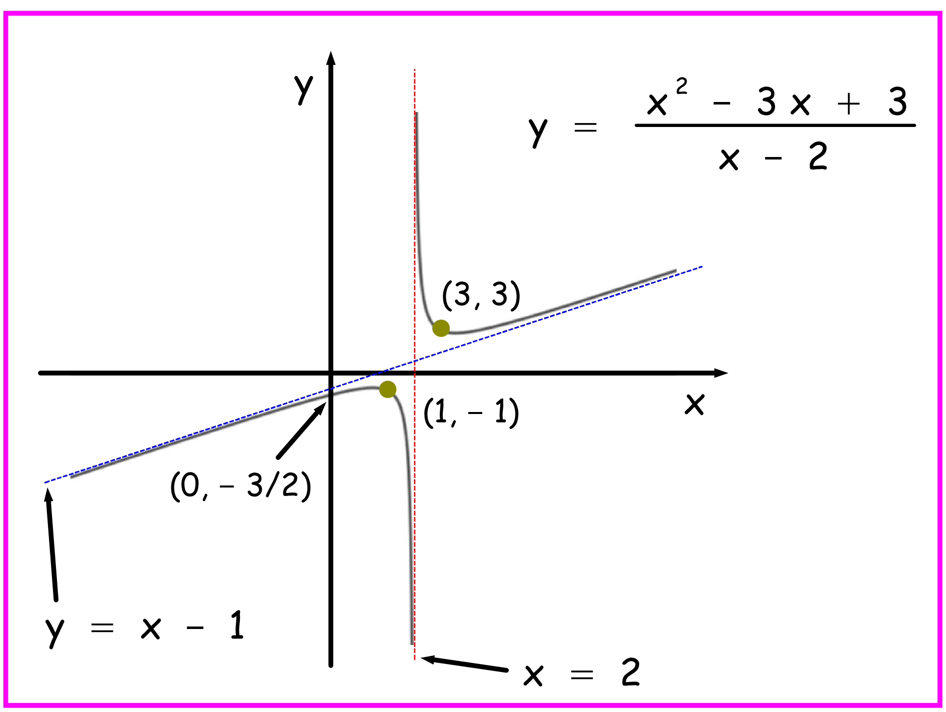
$(1, -1)$ is a local max. ; $(3, 3)$ is a local min.

## Inflexion Points

Attempting to solve $y'' = 0$ gives,

$$\frac{2}{(x - 2)^3} = 0 \Rightarrow 2 = 0$$

which is clearly nonsense. Hence, no inflexion points.



AH Maths - MiA ($2^{nd}$ Edn.)

- pg. 77 Ex. 5.11 Q 1 a, b, c, d, e, g, h, i, j, l.

Ex. 5.11

**1** Sketch these graphs.

a $y = \dfrac{1}{x + 3}$

b $y = \dfrac{3}{2x + 8}$

c $y = \dfrac{x}{x + 2}$

d $y = \dfrac{x - 1}{x + 1}$

e $y = \dfrac{1 - x}{1 + x}$

g $y = \dfrac{x}{(x - 1)(x + 1)}$

h $y = \dfrac{x^2}{x + 1}$

i $y = x - \dfrac{1}{x}$

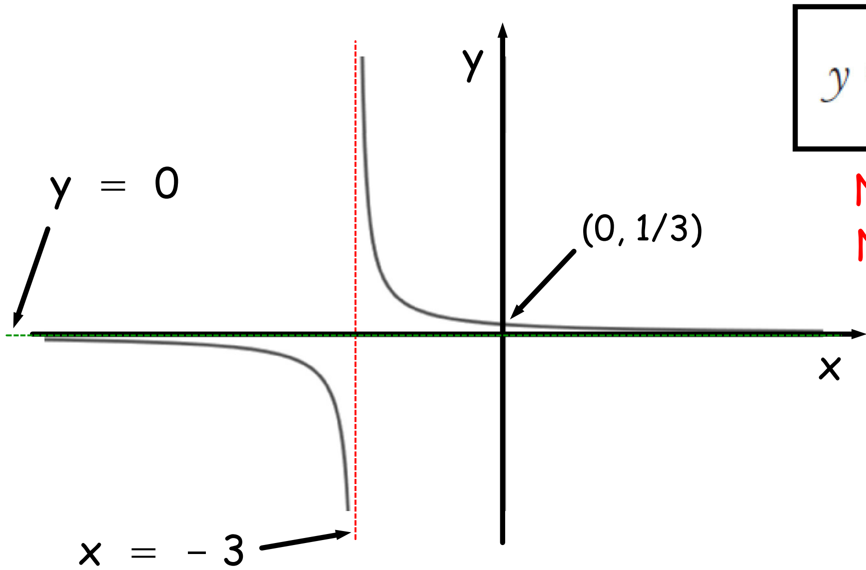
j $y = \dfrac{x^2}{1 - x}$

l $y = \dfrac{1}{(x - 2)(x - 4)}$

Jun 19-11:45

Answers to AH Maths (MiA), pg. 77, Ex. 5.11

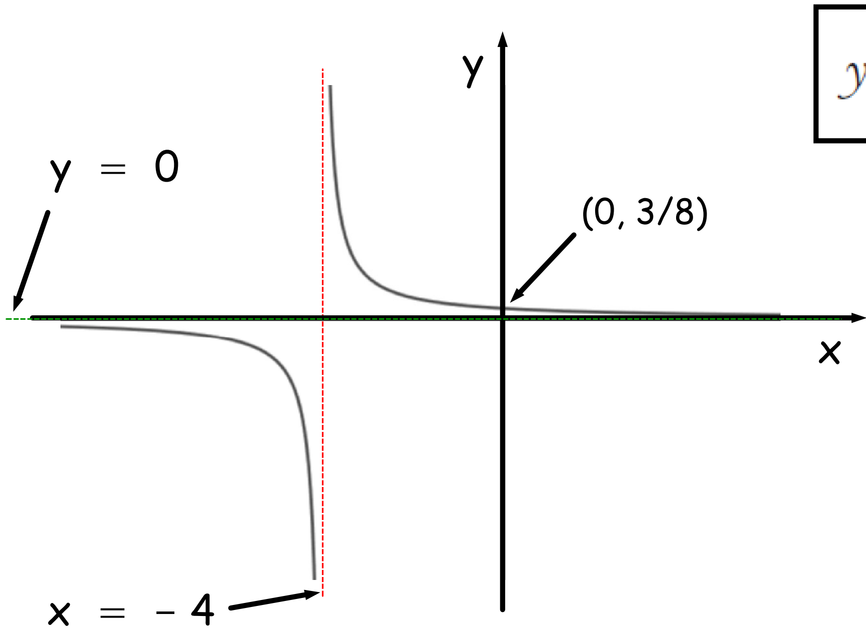
1) (a)




$$y = \frac{1}{x+3}$$

No SP
No P of I

(b)

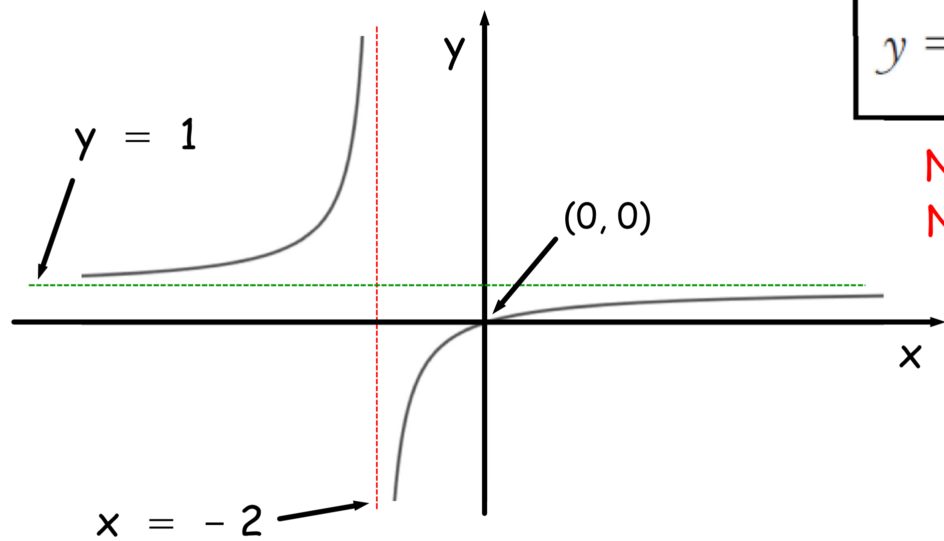



$$y = \frac{3}{2x+8}$$

No SP
No P of I

Jun 1-11:51

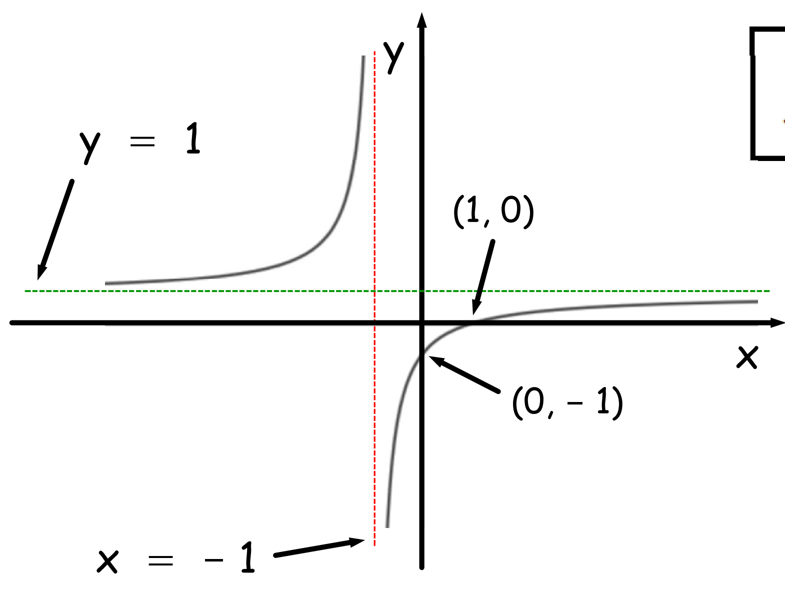
(c)




$$\boxed{y = \frac{x}{x+2}}$$

No SP
No P of I

(d)

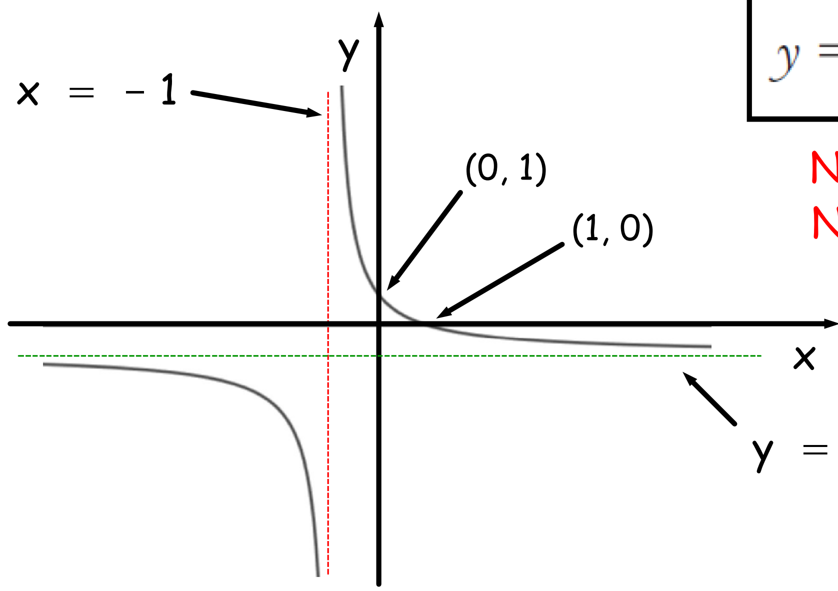



$$\boxed{y = \frac{x-1}{x+1}}$$

No SP
No P of I

Jun 1-11:51

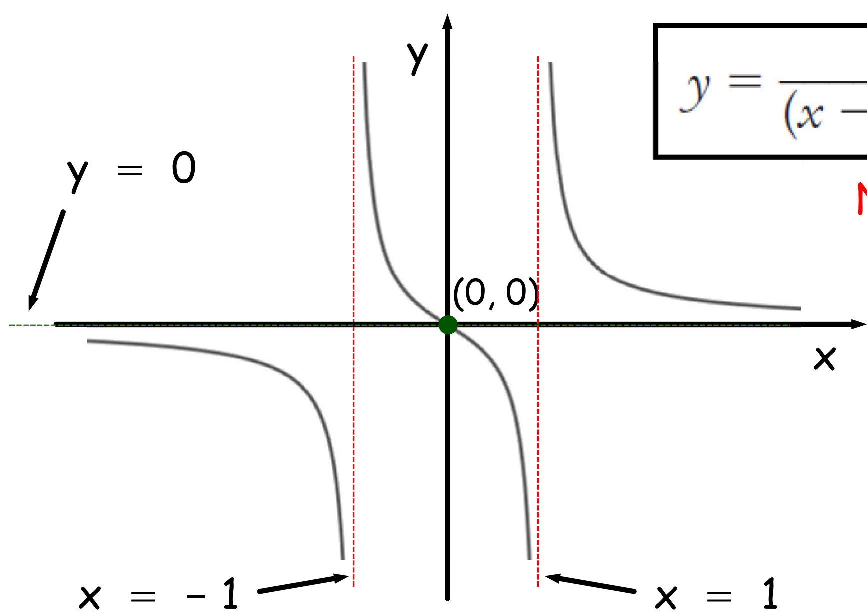
(e)




$$\boxed{y = \frac{1-x}{1+x}}$$

No SP
No P of I

(g)

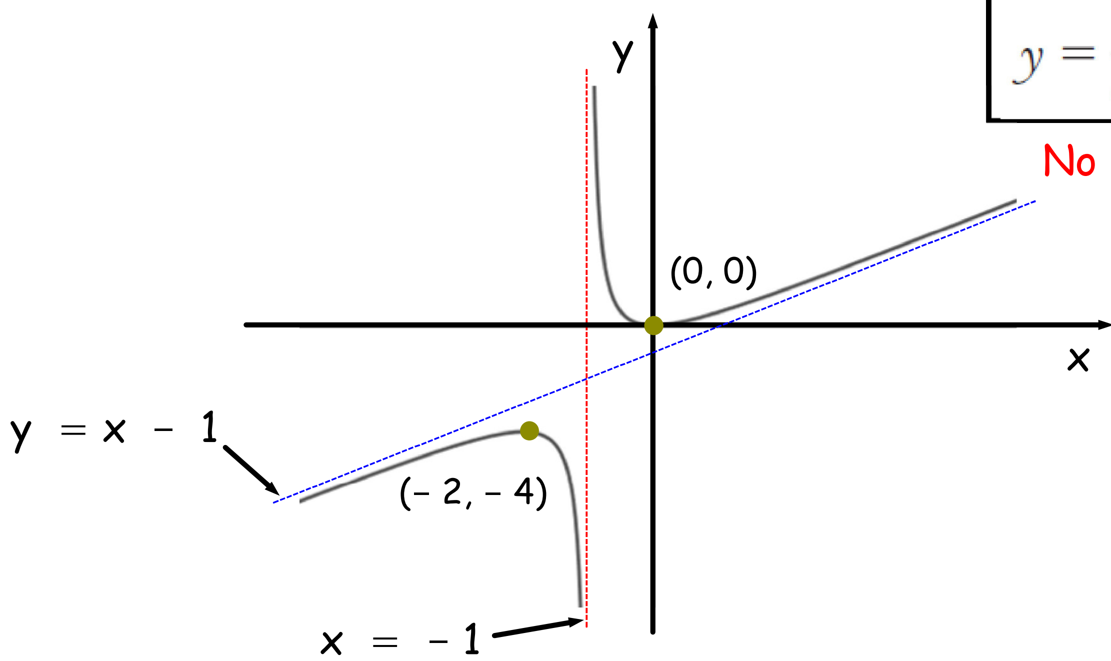



$$\boxed{y = \frac{x}{(x-1)(x+1)}}$$

No SP

Jun 1-11:51

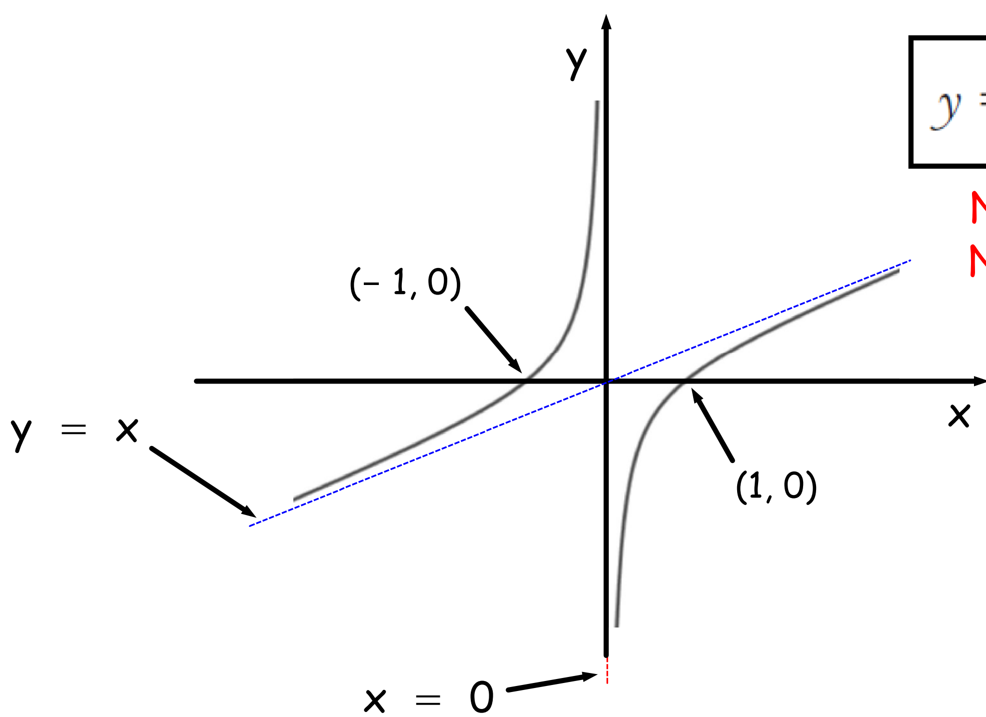
(h)




$$y = \frac{x^2}{x+1}$$

No P of I

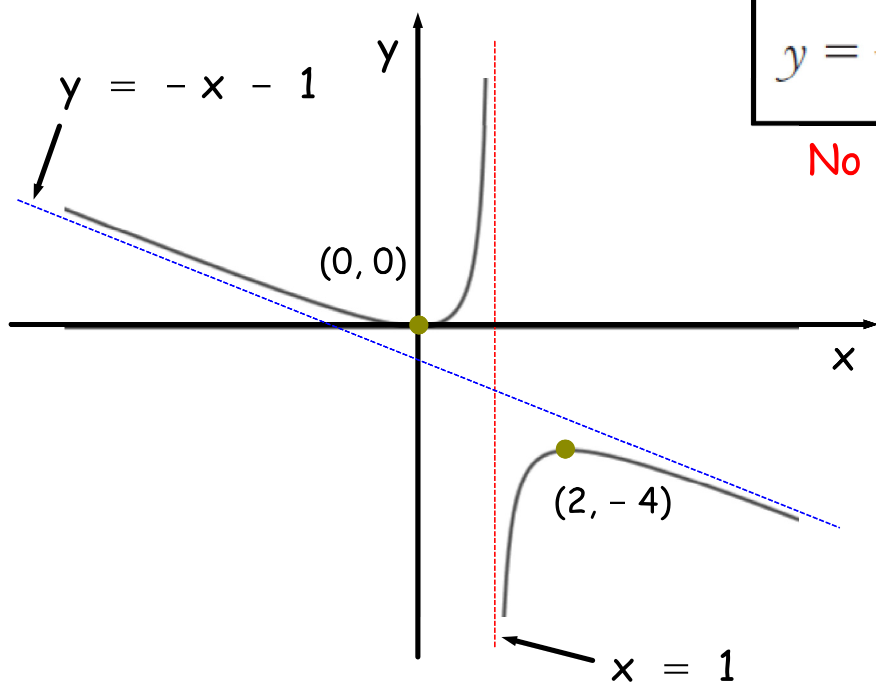
(i)




$$y = x - \frac{1}{x}$$

No SP
No P of I

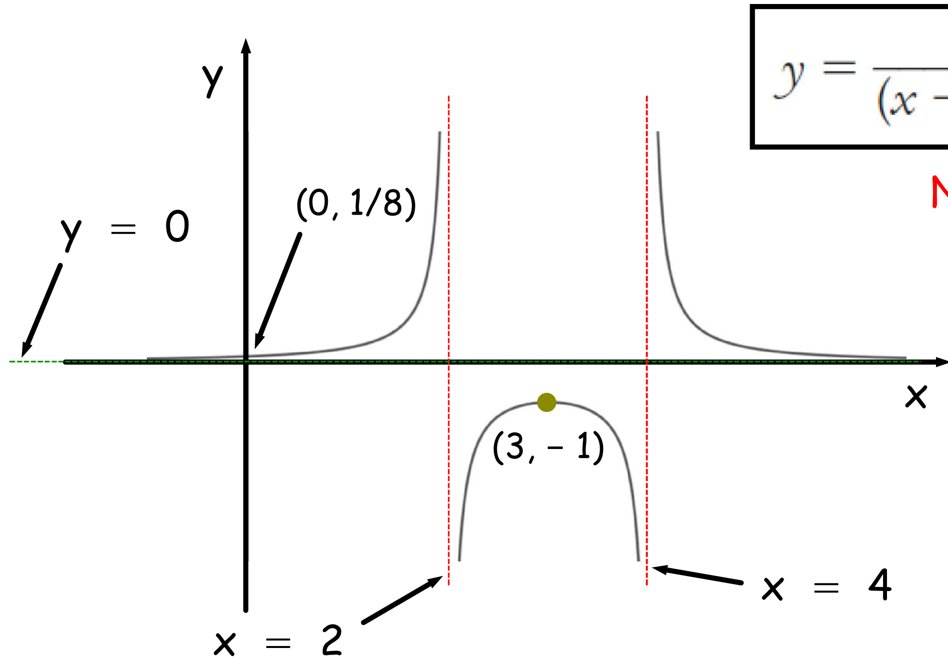
(j)




$$y = \frac{x^2}{1 - x}$$

No P of I

(l)




$$y = \frac{1}{(x-2)(x-4)}$$

No P of I