21 / 12 / 17

Unit 2 : Applications of Calculus - Lesson 2

Areas and Volumes of Revolution

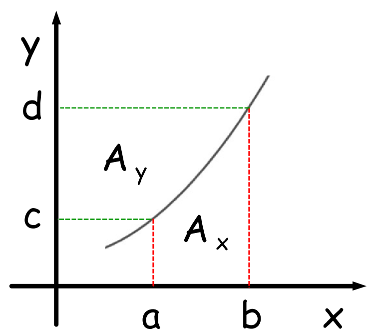
LI

- Work out areas between curves relative to the y - axis.
- Work out volumes of solids of revolution about both axes.

SC

- Integration.

Areas



$y = f(x)$

$\updownarrow$

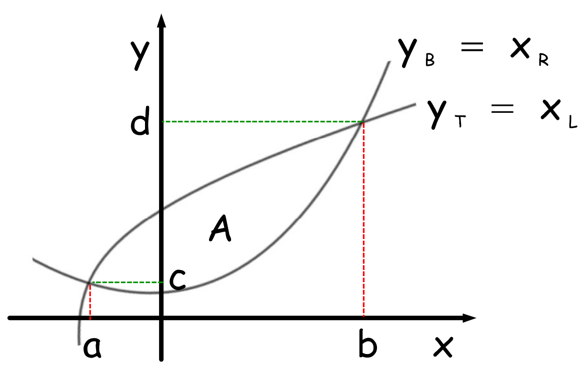
$x = g(y)$

$$A_x = \int_a^b f(x)\, dx$$

Area between $y = f(x)$, the x - axis and the lines $x = a$ and $x = b$

$$A_y = \int_c^d g(y)\, dy$$

Area between $x = g(y)$, the y - axis and the lines $y = c$ and $y = d$



$$A = \int_a^b (y_T - y_B)\, dx$$

('Top – Bottom')

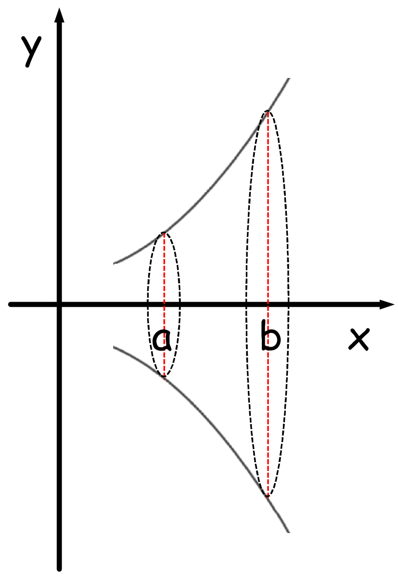
$$A = \int_c^d (x_R - x_L)\, dy$$

('Right – Left')

Dec 20-14:59

Volumes

A volume of solid of revolution is a 3D shape formed by rotating a curve about the x - axis or the y - axis



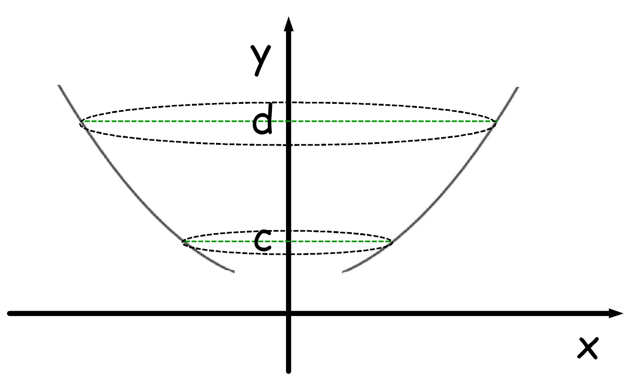
$y = f(x)$

$\updownarrow$

$x = g(y)$

$$V_x = \pi \int_a^b y^2 \, dx$$

Volume about x - axis and the lines $x = a$ and $x = b$



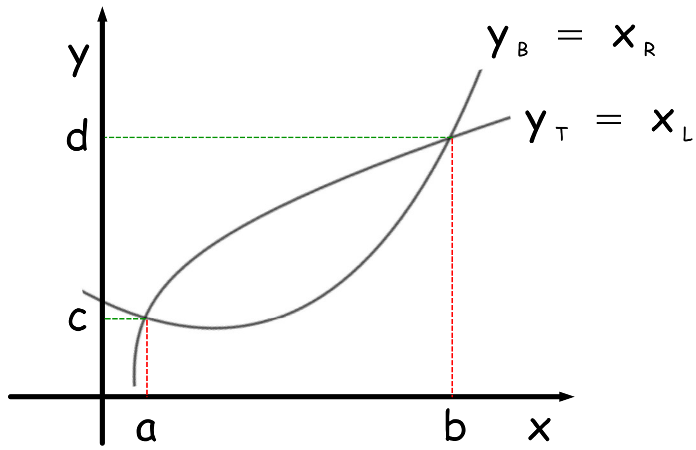
$y = f(x)$

$\updownarrow$

$x = g(y)$

$$V_y = \pi \int_c^d x^2 \, dy$$

Volume about y - axis and the lines $y = c$ and $y = d$



$$V_x = \pi \int_a^b (y_T^2 - y_B^2)\, dx \qquad (\text{'Top - Bottom'})$$

$$V_y = \pi \int_c^d (x_R^2 - x_L^2)\, dy \qquad (\text{'Right - Left'})$$

Dec 21-01:10

Example 1

Find the area enclosed between the curves $x = (1/2)\,y$ and $x = \sqrt{y}$.

The graphs meet when,

$$(1/2)\,y = \sqrt{y}$$

$$(1/4)\,y^2 = y$$

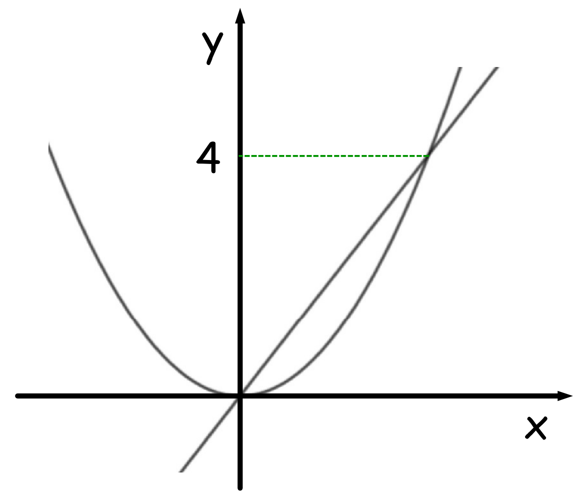
$$y^2 - 4y = 0$$

$$y(y - 4) = 0$$

$$\underline{y = 0,\ y = 4}$$

$x = \sqrt{y} \longleftrightarrow y = x^2$

$x = (1/2)\,y \longleftrightarrow y = 2x$



$$A = \int_0^4 \left(y^{1/2} - (1/2)\,y\right)\,dy$$

$$\therefore \quad A = \left[(2/3)\,y^{3/2} - (1/4)\,y^2\right]_0^4$$

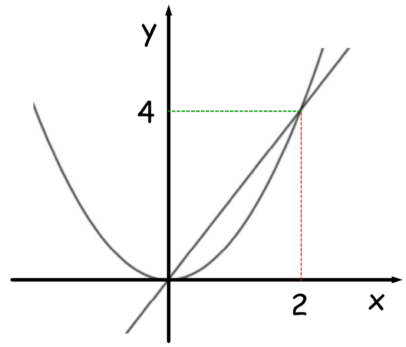
$$\Rightarrow \quad A = ((2/3)(4)(2) - (1/4)\,4^2) - (0 - 0)$$

$$\Rightarrow \quad \boxed{A = 4/3 \text{ square units}}$$

Example 2

Find the volume of solid of revolution formed by rotating the region enclosed between the curves $y = x^2$ and $y = 2x$ 360° about the (a) x - axis (b) y - axis.

$x = \sqrt{y}$ $\updownarrow$ $y = x^2$ $x = (1/2)\,y$ $\updownarrow$ $y = 2x$



(a)

$$V_x = \pi \int_0^2 (4x^2 - x^4)\, dx$$

$$\therefore \quad V_x = \pi \left[(4/3)\,x^3 - (1/5)\,x^5 \right]_0^2$$

$$\Rightarrow \quad V_x = \pi\,((32/3) - (32/5)) - \pi\,(0 - 0)$$

$$\Rightarrow \quad \boxed{V_x = 64\pi/15 \text{ cubic units}}$$

(b)

$$V_y = \pi \int_0^4 (y - (1/4)\,y^2)\, dy$$

$$\therefore \quad V_y = \pi \left[(1/2)\,y^2 - (1/12)\,y^3 \right]_0^4$$

$$\Rightarrow \quad V_y = \pi\,(8 - (64/12)) - \pi\,(0 - 0)$$

$$\Rightarrow \quad \boxed{V_y = 8\pi/3 \;\; u^3}$$

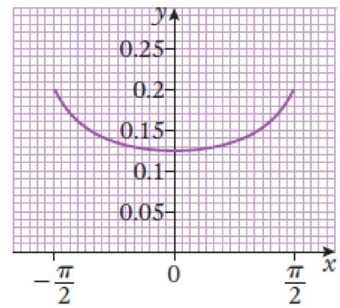
Jun 8-12:56

AH Maths - MiA (2^{nd} Edn.)

- pg. 120-3 Ex. 7.10
  Q 1, 6, 7, 9, 11 a, b, 12 a - c, d (i), 13, 14.

Ex. 7.10

1 The chart shows $y = \dfrac{1}{4\sqrt{4-x^2}}$ in the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
Calculate the area trapped between the curve and the x-axis in this interval.

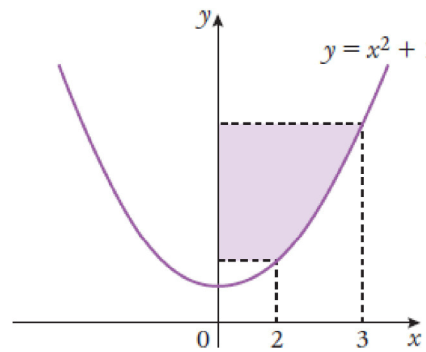


6 The sketch shows $y = x^2 + 1$ close to the origin.

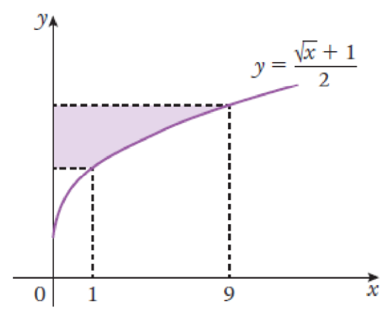
a State the y-values corresponding to $x = 2$ and $x = 3$.

b Express x in terms of y, considering the positive root only.

c Calculate the shaded area.

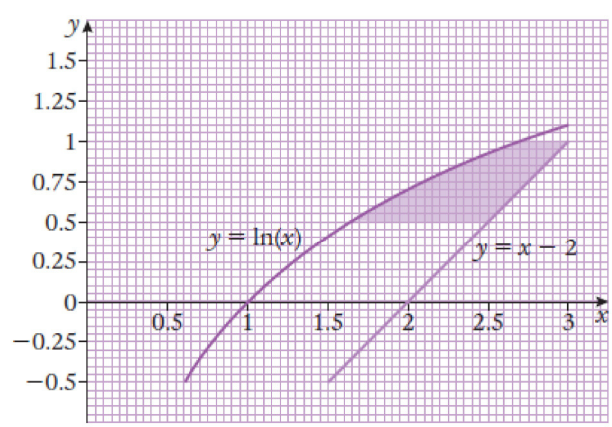


7 Calculate the area trapped between the function $y = \dfrac{\sqrt{x}+1}{2}$ and the y-axis in the interval $1 \le x \le 9$.

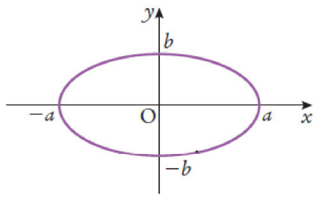


9 The sketch shows $y = \ln x$ and $y = x - 2$ close to the origin.

Calculate the shaded area trapped between $y = \ln x$, $y = x - 2$, $y = 1$ and $y = 0.5$.



11 The equation of an ellipse, centre the origin, and major axis lying on the x-axis, is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (see diagram for explanation of a and b).



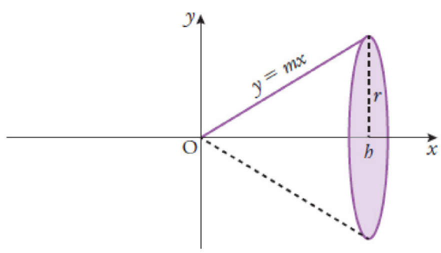
If the shape is rotated about the x-axis, the solid generated is called a *prolate* spheroid. Each cross-section perpendicular to the x-axis is a circle.
In theory, a rugby ball takes this shape.

a Find a formula for the volume of the prolate spheroid using $V = \int_{-a}^{a} \pi y^2 \, dx$.

b If, instead, the shape is rotated about the y-axis, the shape generated is called an *oblate* spheroid. Your geography teacher will tell you the Earth is an oblate spheroid.

Find a formula for the volume of the oblate spheroid using $V = \int_{-b}^{b} \pi x^2 \, dy$.

12 a When a line with equation $y = mx$ is rotated about the x-axis in the interval $0 \le x \le h$, a cone is generated. When $x = h$, $y = r$. $\left[\text{That is, } m = \frac{r}{h}.\right]$

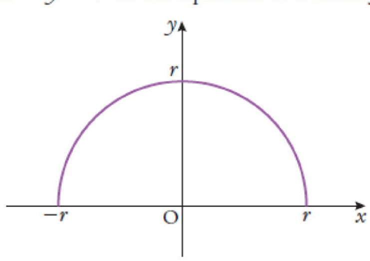


Use integration to find a formula in r and h for the volume of a cone.

b When the line is rotated about the x-axis in the interval $h_1 \le x \le h_2$, a truncated cone is generated. Use integration to find its volume.

c When the line $y = r$, where r is a constant, is rotated about the x-axis in the interval $0 \le x \le h$, a cylinder of height h and radius r is generated.
Use integration to find a formula for its volume.

d $x^2 + y^2 = r^2$ is the equation of a circle. $y = \sqrt{r^2 - x^2}$ is the equation of the semicircle shown.



i If this is rotated about the x-axis, a sphere is generated. Use integration to find a formula for its volume.

13 a Draw a sketch of the curve $y = x^2 + 2x - 3$ in the interval $0 \le x \le 2$.

b Express the equation in the form $y = (x + a)^2 + b$.

c Hence find an expression for x^2 in terms of y.

d The curve is rotated around the y-axis. What is the volume of the shape generated?

14 The curve $y = \sin(x^2)$ is rotated about the y-axis in the interval $0 \le x \le \sqrt{\frac{\pi}{2}}$.

Find the volume of the solid generated.

Answers to AH Maths (MiA), pg. 120-3, Ex. 7.10

1) $(1/2)\sin^{-1}(\pi/4) \approx 0.452$.

6) $38/3$.

7) $13/3$.

9) 0.3054.

11) (a) $(4/3)\pi a b^2$.

(b) $(4/3)\pi a^2 b$.

12) (a) $(1/3)\pi r^2 h$.

(b) $(1/3)\pi r^2 (h_2 - h_1)$.

(c) $\pi r^2 h$.

(d) (i) $(4/3)\pi r^3$.

13) (a) Parabola passing through $(0, -3)$, $(1, 0)$ and $(2, 5)$ - RHS of parabola.

(b) $y = (x + 1)^2 - 4$.

(c) $x^2 = (\sqrt{y + 4} - 1)^2$.

(d) $(40\pi/3)$.

14) $\pi/2\,(\pi - 2)$.