

# AH Mathematics

## Applications of Algebra and Calculus

### Practice Assessment

3

# Solutions

## Applications of Algebra and Calculus Assessment Standard 1.1

1 The Binomial Theorem gives,

$$\begin{aligned}
 (4x + 3)^5 &= \sum_{r=0}^5 \binom{5}{r} (4x)^{5-r} 3^r \\
 \Rightarrow (4x + 3)^5 &= \binom{5}{0} (4x)^5 3^0 + \binom{5}{1} (4x)^4 3^1 + \binom{5}{2} (4x)^3 3^2 \\
 &\quad + \binom{5}{3} (4x)^2 3^3 + \binom{5}{4} (4x)^1 3^4 + \binom{5}{5} (4x)^0 3^5 \\
 \Rightarrow (4x + 3)^5 &= 1 \cdot 1024x^5 \cdot 1 + 5 \cdot 256x^4 \cdot 3 + 10 \cdot 64x^3 \cdot 9 \\
 &\quad + 10 \cdot 16x^2 \cdot 27 + 5 \cdot 4x \cdot 81 + 1 \cdot 1 \cdot 243 \\
 \Rightarrow (4x + 3)^5 &= 1024x^5 + 3840x^4 + 5760x^3 + 4320x^2 + 1620x + 243
 \end{aligned}$$

2  $z_1 = p + i$  and  $z_2 = 3 - 5i$ .

$$\begin{aligned}
 \text{a)} \quad z_1 z_2 &= (p + i)(3 - 5i) \\
 \Rightarrow z_1 z_2 &= 3p + 3i - 5pi - 5i^2 \\
 \Rightarrow z_1 z_2 &= 3p + 3i - 5pi + 5 \\
 \Rightarrow z_1 z_2 &= (3p + 5) + (3 - 5p)i
 \end{aligned}$$

$$\begin{aligned}
 \text{b)} \quad \frac{z_1}{z_2} &= \frac{p + i}{3 - 5i} \quad \times (3 + 5i) \\
 \Rightarrow \frac{z_1}{z_2} &= \frac{(p + i)(3 + 5i)}{(3 - 5i)(3 + 5i)} \\
 \Rightarrow \frac{z_1}{z_2} &= \frac{3p + 3i + 5pi + 5i^2}{9 + 25} \\
 \Rightarrow \frac{z_1}{z_2} &= \left( \frac{3p - 5}{34} \right) + \left( \frac{3 + 5p}{34} \right) i
 \end{aligned}$$

## Applications of Algebra and Calculus Assessment Standard 1.2

3       $7, 25, 43, \dots$ ;  $a = 7, d = 18$ .

$$\text{a)} \quad u_n = a + (n - 1)d$$

$$\therefore u_{16} = 7 + (16 - 1).18$$

$$\Rightarrow u_{16} = 277$$

$$\text{b)} \quad S_n = \frac{n}{2} (2a + (n - 1)d)$$

$$\therefore S_{16} = \frac{16}{2} (2(7) + (16 - 1).18)$$

$$\Rightarrow S_{16} = 2272$$

4       $4, 36, 324, \dots$ ;  $a = 4, r = 9$ .

$$\text{a)} \quad u_n = a r^{n-1}$$

$$\therefore u_5 = 4 \cdot 9^{5-1}$$

$$\Rightarrow u_5 = 26244$$

$$\text{b)} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\therefore S_5 = \frac{4(1 - 9^5)}{1 - 9}$$

$$\Rightarrow S_5 = 29524$$

5       $f(x) = e^{6x}$ .

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\therefore e^{6x} = 1 + (6x) + \frac{(6x)^2}{2!} + \frac{(6x)^3}{3!} + \dots$$

$$\Rightarrow e^{6x} = 1 + 6x + \frac{36x^2}{2} + \frac{216x^3}{6} + \dots$$

$$\Rightarrow e^{6x} = 1 + 6x + 18x^2 + 36x^3 + \dots$$

## Applications of Algebra and Calculus Assessment Standard 1.3

6

$$\begin{aligned}
 & \sum_{k=1}^{20} (6k - 5) = \left( \sum_{k=1}^{20} 6k \right) - \left( \sum_{k=1}^{20} 5 \right) \\
 \Rightarrow & \sum_{k=1}^{20} (6k - 5) = 6 \left( \sum_{k=1}^{20} k \right) - 5 \left( \sum_{k=1}^{20} 1 \right) \\
 \Rightarrow & \sum_{k=1}^{20} (6k - 5) = 6 \left( \frac{1}{2} \cdot 20(20 + 1) \right) - 5(20) \\
 \Rightarrow & \sum_{k=1}^{20} (6k - 5) = 60(21) - 100 \\
 \Rightarrow & \sum_{k=1}^{20} (6k - 5) = 1160
 \end{aligned}$$

7       $P(n) : \sum_{r=1}^n 10r = 5n(n + 1)$

Base Case :

$$LHS = \sum_{r=1}^1 10r = 10 \cdot 1 = 10$$

$$RHS = 5 \cdot 1 \cdot (1 + 1) = 10$$

As  $10 = 10$ ,  $LHS = RHS \Rightarrow P(1)$  is true

Inductive Step :

Assume  $P(k)$  is true for some  $k \in \mathbb{N}$ , i.e.  $\sum_{r=1}^k 10r = 5k(k + 1)$

Required To Prove :  $\sum_{r=1}^{k+1} 10r = 5(k+1)(k+2)$

$$\begin{aligned}\sum_{r=1}^{k+1} 10r &= \left( \sum_{r=1}^k 10r \right) + 10(k+1) \\ \Rightarrow \sum_{r=1}^{k+1} 10r &= 5k(k+1) + 10(k+1) \\ \Rightarrow \sum_{r=1}^{k+1} 10r &= 5(k+1)(k+2); \text{ hence } P(k) \text{ true } \Rightarrow P(k+1) \text{ true}\end{aligned}$$

As  $P(1)$  is true and  $P(k)$  true  $\Rightarrow P(k+1)$  true,  
the PMI implies that  $P(n)$  is true  $\forall n \in \mathbb{N}$

### Applications of Algebra and Calculus Assessment Standard 1.4

8  $f(x) = \frac{x^2 - 2x - 1}{x - 4}, x \in \mathbb{R} : x \neq 4$

a) Vertical asymptote :  $x = 4$

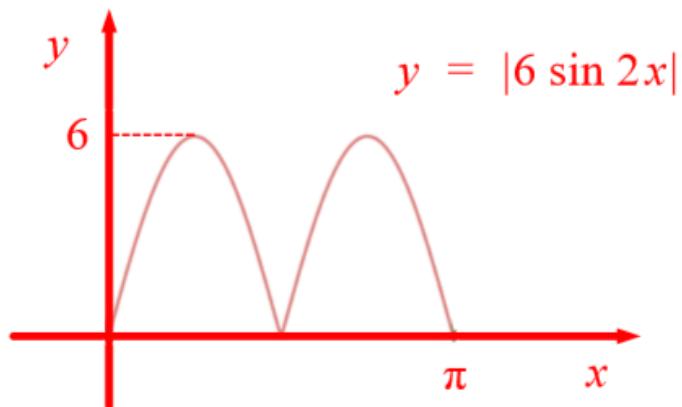
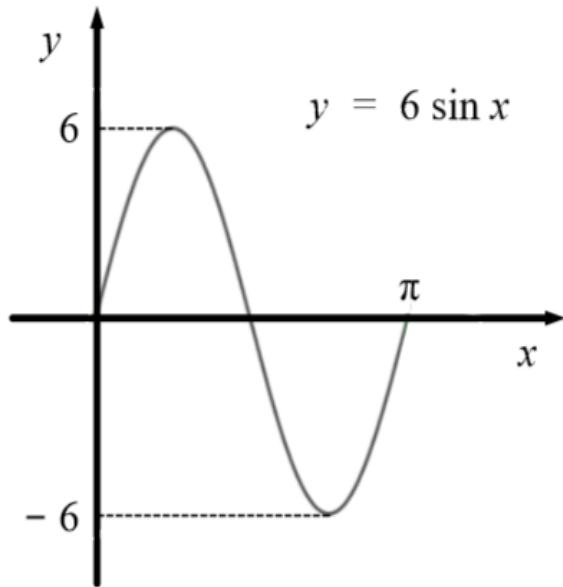
b)

$$\begin{array}{r} x + 2 \\ x - 4 \quad \overline{)x^2 - 2x - 1} \\ \underline{x^2 - 4x} \\ \hline 2x - 1 \\ \underline{x - 8} \\ \hline 7 \end{array}$$

$$\begin{aligned}\therefore f(x) &= x + 2 + \frac{7}{x-4} = g(x) + \frac{7}{x-4}; \\ g(x) \neq 0 &\Rightarrow \text{non-vertical asymptote}\end{aligned}$$

Non-vertical asymptote :  $y = x + 2$

9



$$10 \quad y = f(x) = 6x^3 + x$$

$$\therefore f'(x) = 18x^2 + 1$$

$$\Rightarrow f''(x) = 36x$$

For a POI,  $f''(x) = 0 \Rightarrow 36x = 0 \Rightarrow x = 0$

If  $x$  is just smaller than 0,  $f''(x) < 0$  ( $f$  is concave down); if  $x$  is just bigger than 0,  $f''(x) > 0$  ( $f$  is concave up).

**As there is a change in concavity at  $x = 0$ ,  $f$  has a POI at  $x = 0$**

## Applications of Algebra and Calculus Assessment Standard 1.5

11       $v(t) = \frac{400t}{7t + 15}$

$$a(t) = v'(t)$$

$$\therefore a(t) = \frac{400(7t + 15) - 7(400t)}{(7t + 15)^2}$$

$$\Rightarrow a(t) = \frac{6000}{(7t + 15)^2}$$

$$\therefore a(5) = \frac{6000}{(7(5) + 15)^2}$$

$$\Rightarrow a(5) = \frac{12}{5} \quad (= 12.4) \text{ m s}^{-2}$$

12       $y = \sqrt{1 + \cos 3x}$

$$V = \pi \int_0^{\frac{\pi}{6}} y^2 dx$$

$$\therefore V = \pi \int_0^{\frac{\pi}{6}} (1 + \cos 3x) dx$$

$$\Rightarrow V = \pi \left[ x + \frac{1}{3} \sin 3x \right]_0^{\frac{\pi}{6}}$$

$$\Rightarrow V = \pi \left( \frac{\pi}{6} + \frac{1}{3} \sin \frac{\pi}{2} \right) - \pi \left( 0 + \frac{1}{3} \sin 0 \right)$$

$$\Rightarrow V = \frac{\pi^2}{6} + \frac{\pi}{3} \cdot 1 - \pi \cdot 0 - \frac{\pi}{3} \cdot 0$$

$$\Rightarrow V = \frac{\pi^2}{6} + \frac{\pi}{3}$$

$$\Rightarrow V = \frac{\pi}{6} (\pi + 2) \text{ units}^3$$