

# AH Mathematics

## Applications of Algebra and Calculus

### Practice Assessment

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# Solutions

## Applications of Algebra and Calculus Assessment Standard 1.1

1 The Binomial Theorem gives,

$$\begin{aligned}
 (3x + 2)^5 &= \sum_{r=0}^5 \binom{5}{r} (3x)^{5-r} 2^r \\
 \Rightarrow (3x + 2)^5 &= \binom{5}{0} (3x)^5 2^0 + \binom{5}{1} (3x)^4 2^1 + \binom{5}{2} (3x)^3 2^2 \\
 &\quad + \binom{5}{3} (3x)^2 2^3 + \binom{5}{4} (3x)^1 2^4 + \binom{5}{5} (3x)^0 2^5 \\
 \Rightarrow (3x + 2)^5 &= 1 \cdot 243x^5 \cdot 1 + 5 \cdot 81x^4 \cdot 2 + 10 \cdot 27x^3 \cdot 4 \\
 &\quad + 10 \cdot 9x^2 \cdot 8 + 5 \cdot 3x \cdot 16 + 1 \cdot 1 \cdot 32 \\
 \Rightarrow (3x + 2)^5 &= 243x^5 + 810x^4 + 1080x^3 + 720x^2 + 240x + 32
 \end{aligned}$$

2  $z_1 = p + i$  and  $z_2 = 5 - 2i$ .

a)  $z_1 z_2 = (p + i)(5 - 2i)$

$$\begin{aligned}
 \Rightarrow z_1 z_2 &= 5p + 5i - 2pi - 2i^2 \\
 \Rightarrow z_1 z_2 &= 5p + 5i - 2pi + 2 \\
 \Rightarrow z_1 z_2 &= (5p + 2) + (5 - 2p)i
 \end{aligned}$$

b)  $\frac{z_1}{z_2} = \frac{p+i}{5-2i}$

$$\begin{aligned}
 \Rightarrow \frac{z_1}{z_2} &= \frac{(p+i)(5+2i)}{(5-2i)(5+2i)} \\
 \Rightarrow \frac{z_1}{z_2} &= \frac{5p + 5i + 2pi + 2i^2}{25 + 4} \\
 \Rightarrow \frac{z_1}{z_2} &= \left( \frac{5p - 2}{29} \right) + \left( \frac{5 + 2p}{29} \right) i
 \end{aligned}$$

## Applications of Algebra and Calculus Assessment Standard 1.2

3         $6, 20, 34, \dots ; a = 6, d = 14.$

$$\text{a)} \quad u_n = a + (n - 1)d$$

$$\therefore u_{40} = 6 + (40 - 1).14$$

$$\Rightarrow u_{40} = 552$$

$$\text{b)} \quad S_n = \frac{n}{2} (2a + (n - 1)d)$$

$$\therefore S_{40} = \frac{40}{2} (2(6) + (40 - 1).14)$$

$$\Rightarrow S_{40} = 11160$$

4         $12, 84, 588, \dots ; a = 12, r = 7.$

$$\text{a)} \quad u_n = a r^{n-1}$$

$$\therefore u_7 = 12 \cdot 7^{7-1}$$

$$\Rightarrow u_7 = 1411788$$

$$\text{b)} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\therefore S_7 = \frac{12(1 - 7^7)}{1 - 7}$$

$$\Rightarrow S_7 = 1647084$$

5         $f(x) = e^{3x}.$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\therefore e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots$$

$$\Rightarrow e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \dots$$

$$\Rightarrow e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \dots$$

## Applications of Algebra and Calculus Assessment Standard 1.3

6

$$\begin{aligned}
 \sum_{k=1}^{12} (5k - 4) &= \left( \sum_{k=1}^{12} 5k \right) - \left( \sum_{k=1}^{12} 4 \right) \\
 \Rightarrow \sum_{k=1}^{12} (5k - 4) &= 5 \left( \sum_{k=1}^{12} k \right) - 4 \left( \sum_{k=1}^{12} 1 \right) \\
 \Rightarrow \sum_{k=1}^{12} (5k - 4) &= 5 \left( \frac{1}{2} \cdot 12(12 + 1) \right) - 4(12) \\
 \Rightarrow \sum_{k=1}^{12} (5k - 4) &= 30(13) - 48 \\
 \Rightarrow \sum_{k=1}^{12} (5k - 4) &= 342
 \end{aligned}$$

7       $P(n) : \sum_{r=1}^n 8r = 4n(n + 1)$

Base Case :

$$LHS = \sum_{r=1}^1 8r = 8 \cdot 1 = 8$$

$$RHS = 4 \cdot 1 \cdot (1 + 1) = 8$$

As  $8 = 8$ ,  $LHS = RHS \Rightarrow P(1)$  is true

Inductive Step :

Assume  $P(k)$  is true for some  $k \in \mathbb{N}$ , i.e.  $\sum_{r=1}^k 8r = 4k(k + 1)$

Required To Prove :  $\sum_{r=1}^{k+1} 8r = 4(k+1)(k+2)$

$$\begin{aligned}\sum_{r=1}^{k+1} 8r &= \left( \sum_{r=1}^k 8r \right) + 8(k+1) \\ \Rightarrow \sum_{r=1}^{k+1} 8r &= 4k(k+1) + 8(k+1) \\ \Rightarrow \sum_{r=1}^{k+1} 8r &= 4(k+1)(k+2); \text{ hence } P(k) \text{ true } \Rightarrow P(k+1) \text{ true}\end{aligned}$$

As  $P(1)$  is true and  $P(k)$  true  $\Rightarrow P(k+1)$  true,  
the PMI implies that  $P(n)$  is true  $\forall n \in \mathbb{N}$

### Applications of Algebra and Calculus Assessment Standard 1.4

8  $f(x) = \frac{x^2 + 3x - 14}{x - 3}, x \in \mathbb{R} : x \neq 3$

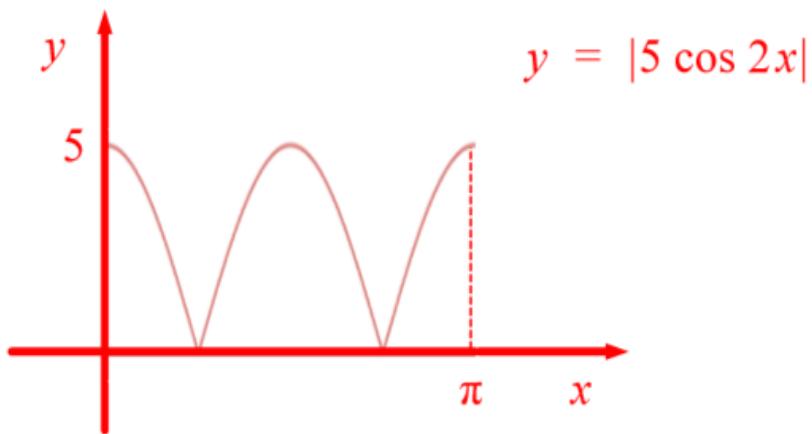
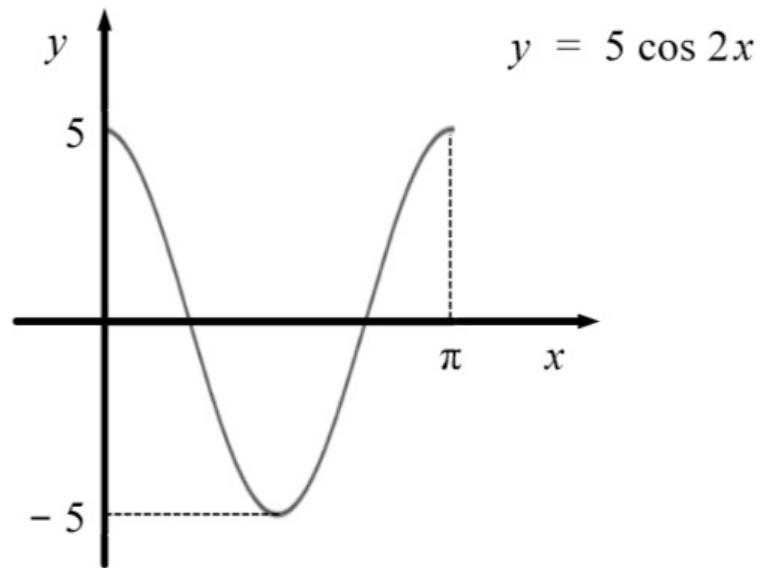
a) Vertical asymptote :  $x = 3$

b)

$$\begin{array}{r} x + 6 \\ x - 3 \quad \overline{)x^2 + 3x - 14} \\ x^2 - 3x \\ \hline 6x - 14 \\ 6x - 18 \\ \hline 4 \end{array}$$

$$\begin{aligned}\therefore f(x) &= x + 6 + \frac{4}{x-3} = g(x) + \frac{4}{x-3}; \\ g(x) \neq 0 &\Rightarrow \text{non-vertical asymptote}\end{aligned}$$

Non-vertical asymptote :  $y = x + 6$



$$10 \quad y = f(x) = 5x^3 - x$$

$$\therefore f'(x) = 15x^2 - 1$$

$$\Rightarrow f''(x) = 30x$$

For a POI,  $f''(x) = 0 \Rightarrow 30x = 0 \Rightarrow x = 0$

If  $x$  is just smaller than 0,  $f''(x) < 0$  ( $f$  is concave down); if  $x$  is just bigger than 0,  $f''(x) > 0$  ( $f$  is concave up).

As there is a change in concavity at  $x = 0$ ,  $f$  has a POI at  $x = 0$

## Applications of Algebra and Calculus Assessment Standard 1.5

$$11 \quad v(t) = \frac{300t}{4t + 9}$$

$$a(t) = v'(t)$$

$$\therefore a(t) = \frac{300(4t + 9) - 4(300t)}{(4t + 9)^2}$$

$$\Rightarrow a(t) = \frac{2700}{(4t + 9)^2}$$

$$\therefore a(2) = \frac{2700}{(4(2) + 9)^2}$$

$$\Rightarrow a(2) = \frac{2700}{289} \ (\approx 9.34) \text{ m s}^{-2}$$

$$12 \quad y = \sqrt{1 + \sin 3x}$$

$$V = \pi \int_0^{\frac{\pi}{3}} y^2 \ dx$$

$$\therefore V = \pi \int_0^{\frac{\pi}{3}} (1 + \sin 3x) \ dx$$

$$\Rightarrow V = \pi \left[ x - \frac{1}{3} \cos 3x \right]_0^{\frac{\pi}{3}}$$

$$\Rightarrow V = \pi \left( \frac{\pi}{3} - \frac{1}{3} \cos \pi \right) - \pi \left( 0 - \frac{1}{3} \cos 0 \right)$$

$$\Rightarrow V = \frac{\pi^2}{3} - \frac{\pi}{3} \cdot (-1) - \pi \cdot 0 + \frac{\pi}{3} \cdot 1$$

$$\Rightarrow V = \frac{\pi^2}{3} + \frac{2\pi}{3}$$

$$\Rightarrow V = \frac{\pi}{3} (\pi + 2) \text{ units}^3$$