

# AH Mathematics

## Applications of Algebra and Calculus

### Practice Assessment

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# Solutions

## Applications of Algebra and Calculus Assessment Standard 1.1

1 The Binomial Theorem gives,

$$\begin{aligned}
 (2x + 3)^5 &= \sum_{r=0}^5 \binom{5}{r} (2x)^{5-r} 3^r \\
 \Rightarrow (2x + 3)^5 &= \binom{5}{0} (2x)^5 3^0 + \binom{5}{1} (2x)^4 3^1 + \binom{5}{2} (2x)^3 3^2 \\
 &\quad + \binom{5}{3} (2x)^2 3^3 + \binom{5}{4} (2x)^1 3^4 + \binom{5}{5} (2x)^0 3^5 \\
 \Rightarrow (2x + 3)^5 &= 1 \cdot 32x^5 \cdot 1 + 5 \cdot 16x^4 \cdot 3 + 10 \cdot 8x^3 \cdot 9 \\
 &\quad + 10 \cdot 4x^2 \cdot 27 + 5 \cdot 2x \cdot 81 + 1 \cdot 1 \cdot 243 \\
 \Rightarrow (2x + 3)^5 &= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243
 \end{aligned}$$

2  $z_1 = p + i$  and  $z_2 = 4 - 3i$ .

a)  $z_1 z_2 = (p + i)(4 - 3i)$

$$\begin{aligned}
 \Rightarrow z_1 z_2 &= 4p + 4i - 3pi - 3i^2 \\
 \Rightarrow z_1 z_2 &= 4p + 4i - 3pi + 3 \\
 \Rightarrow z_1 z_2 &= (4p + 3) + (4 - 3p)i
 \end{aligned}$$

b)  $\frac{z_1}{z_2} = \frac{p+i}{4-3i}$  × (4 + 3i)  
 $\Rightarrow \frac{z_1}{z_2} = \frac{(p+i)(4+3i)}{(4-3i)(4+3i)}$

$$\begin{aligned}
 \Rightarrow \frac{z_1}{z_2} &= \frac{4p + 4i + 3pi + 3i^2}{16 + 9} \\
 \Rightarrow \frac{z_1}{z_2} &= \left( \frac{4p - 3}{25} \right) + \left( \frac{4 + 3p}{25} \right) i
 \end{aligned}$$

## Applications of Algebra and Calculus Assessment Standard 1.2

3         $5, 17, 29, \dots$ ;  $a = 5, d = 12$ .

$$\text{a)} \quad u_n = a + (n - 1)d$$

$$\therefore u_{20} = 5 + (20 - 1).12$$

$$\Rightarrow u_{20} = 233$$

$$\text{b)} \quad S_n = \frac{n}{2} (2a + (n - 1)d)$$

$$\therefore S_{20} = \frac{20}{2} (2(5) + (20 - 1).12)$$

$$\Rightarrow S_{20} = 2380$$

4         $6, 30, 150, \dots$ ;  $a = 6, r = 5$ .

$$\text{a)} \quad u_n = a r^{n-1}$$

$$\therefore u_9 = 6 \cdot 5^{9-1}$$

$$\Rightarrow u_9 = 2343750$$

$$\text{b)} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\therefore S_9 = \frac{6(1 - 5^9)}{1 - 5}$$

$$\Rightarrow S_9 = 2929686$$

5         $f(x) = e^{4x}$ .

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\therefore e^{4x} = 1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \dots$$

$$\Rightarrow e^{4x} = 1 + 4x + \frac{16x^2}{2} + \frac{64x^3}{6} + \dots$$

$$\Rightarrow e^{4x} = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \dots$$

## Applications of Algebra and Calculus Assessment Standard 1.3

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$$\begin{aligned}
 \sum_{k=1}^{14} (4k - 3) &= \left( \sum_{k=1}^{14} 4k \right) - \left( \sum_{k=1}^{14} 3 \right) \\
 \Rightarrow \sum_{k=1}^{14} (4k - 3) &= 4 \left( \sum_{k=1}^{14} k \right) - 3 \left( \sum_{k=1}^{14} 1 \right) \\
 \Rightarrow \sum_{k=1}^{14} (4k - 3) &= 4 \left( \frac{1}{2} \cdot 14(14 + 1) \right) - 3(14) \\
 \Rightarrow \sum_{k=1}^{14} (4k - 3) &= 28(15) - 42 \\
 \Rightarrow \sum_{k=1}^{14} (4k - 3) &= 378
 \end{aligned}$$

7       $P(n) : \sum_{r=1}^n 4r = 2n(n + 1)$

Base Case :

$$LHS = \sum_{r=1}^1 4r = 4 \cdot 1 = 4$$

$$RHS = 2 \cdot 1 \cdot (1 + 1) = 4$$

As  $4 = 4$ ,  $LHS = RHS \Rightarrow P(1)$  is true

Inductive Step :

Assume  $P(k)$  is true for some  $k \in \mathbb{N}$ , i.e.  $\sum_{r=1}^k 4r = 2k(k + 1)$

Required To Prove :  $\sum_{r=1}^{k+1} 4r = 2(k+1)(k+2)$

$$\begin{aligned}\sum_{r=1}^{k+1} 4r &= \left( \sum_{r=1}^k 4r \right) + 4(k+1) \\ \Rightarrow \sum_{r=1}^{k+1} 4r &= 2k(k+1) + 4(k+1) \\ \Rightarrow \sum_{r=1}^{k+1} 4r &= 2(k+1)(k+2); \text{ hence } P(k) \text{ true } \Rightarrow P(k+1) \text{ true}\end{aligned}$$

As  $P(1)$  is true and  $P(k)$  true  $\Rightarrow P(k+1)$  true,  
the PMI implies that  $P(n)$  is true  $\forall n \in \mathbb{N}$

### Applications of Algebra and Calculus Assessment Standard 1.4

8  $f(x) = \frac{x^2 - x + 3}{x - 2}, x \in \mathbb{R} : x \neq 2$

a) Vertical asymptote :  $x = 2$

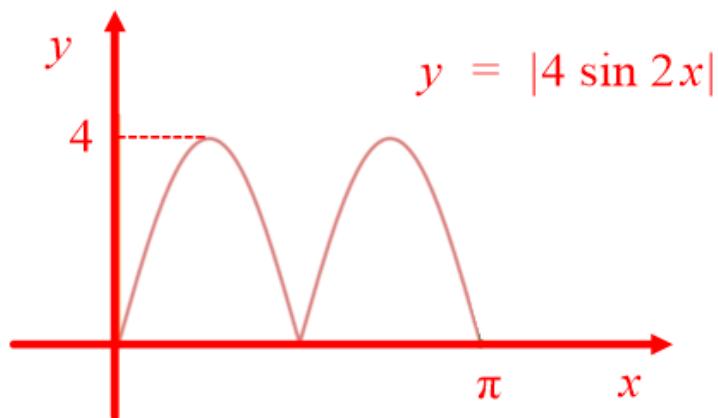
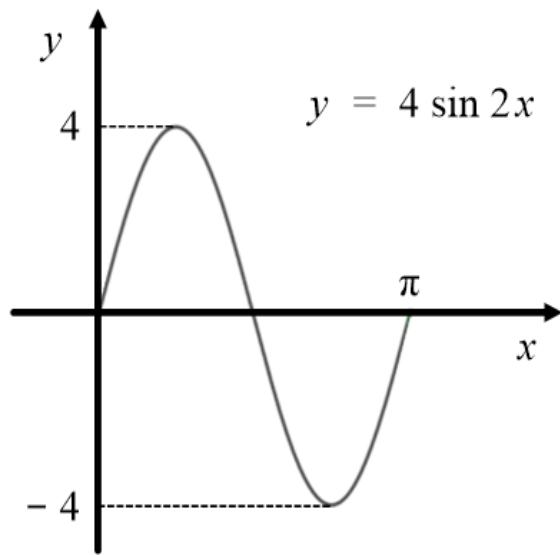
b)

$$\begin{array}{r} x + 1 \\ x - 2 \overline{) x^2 - x + 3} \\ x^2 - 2x \\ \hline x + 3 \\ x - 2 \\ \hline 5 \end{array}$$

$$\begin{aligned}\therefore f(x) &= x + 1 + \frac{5}{x-2} = g(x) + \frac{5}{x-2}; \\ g(x) \neq 0 &\Rightarrow \text{non-vertical asymptote}\end{aligned}$$

Non-vertical asymptote :  $y = x + 1$

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$$10 \quad y = f(x) = 4x^3 + x$$

$$\therefore f'(x) = 12x^2 + 1$$

$$\Rightarrow f''(x) = 24x$$

$$\text{For a POI, } f''(x) = 0 \Rightarrow 24x = 0 \Rightarrow x = 0$$

If  $x$  is just smaller than 0,  $f''(x) < 0$  ( $f$  is concave down); if  $x$  is just bigger than 0,  $f''(x) > 0$  ( $f$  is concave up).

As there is a change in concavity at  $x = 0$ ,  $f$  has a POI at  $x = 0$

## Applications of Algebra and Calculus Assessment Standard 1.5

$$11 \quad v(t) = \frac{200t}{3t + 13}$$

$$a(t) = v'(t)$$

$$\therefore a(t) = \frac{200(3t + 13) - 3(200t)}{(3t + 13)^2}$$

$$\Rightarrow a(t) = \frac{2600}{(3t + 13)^2}$$

$$\therefore a(3) = \frac{2600}{(3(3) + 13)^2}$$

$$\Rightarrow a(3) = \frac{650}{121} \ (\approx 5.37) \text{ m s}^{-2}$$

$$12 \quad y = \sqrt{1 + \cos 2x}$$

$$V = \pi \int_0^{\frac{\pi}{2}} y^2 \ dx$$

$$\therefore V = \pi \int_0^{\frac{\pi}{2}} (1 + \cos 2x) \ dx$$

$$\Rightarrow V = \pi \left[ x + \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow V = \pi \left( \frac{\pi}{2} + \frac{1}{2} \sin \pi \right) - \pi \left( 0 + \frac{1}{2} \sin 0 \right)$$

$$\Rightarrow V = \frac{\pi^2}{2} + \frac{\pi}{2} \cdot 0 - \pi \cdot 0 - \frac{\pi}{2} \cdot 0$$

$$\Rightarrow V = \frac{\pi^2}{2} \text{ units}^3$$