2007 Applied Mathematics

Advanced Higher – Numerical Analysis

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question B6, M1 means a method mark for using the partial fractions to work out the are. E is shorthand for error. For example, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Applied Mathematics 2007
Section A – Numerical Analysis

A1. $f(x) = \ln(7 - 3x) \qquad f'(x) = \dfrac{-3}{7 - 3x} \qquad f''(x) = \dfrac{-9}{(7 - 3x)^2}$ **1,1**

Taylor polynomial is $p(2 + h) = 0 - 3h - \frac{9}{2}h^2$. **1**

For $f(1\cdot97)$, $h = -0\cdot03$ and
$p(1\cdot97) = 0 + 0\cdot09 - 0\cdot00405 = 0\cdot08595$. **2**

$f(x)$ should be fairly sensitive to small changes in x as coefficients are > 1. **1**

A2.

$$L(1\cdot3) = \frac{(1\cdot3-0\cdot7)(1\cdot3-1\cdot0)(1\cdot3-2\cdot0)}{(0\cdot0-0\cdot7)(0\cdot0-1\cdot0)(0\cdot0-2\cdot0)}2\cdot716 + \frac{(1\cdot3-0\cdot0)(1\cdot3-1\cdot0)(1\cdot3-2\cdot0)}{(0\cdot7-0\cdot0)(0\cdot7-1\cdot0)(0\cdot7-2\cdot0)}2\cdot315$$

$$+ \frac{(1\cdot3-0\cdot0)(1\cdot3-0\cdot7)(1\cdot3-2\cdot0)}{(1\cdot0-0\cdot0)(1\cdot0-0\cdot7)(1\cdot0-2\cdot0)}2\cdot103 + \frac{(1\cdot3-0\cdot0)(1\cdot3-0\cdot7)(1\cdot3-1\cdot0)}{(2\cdot0-0\cdot0)(2\cdot0-0\cdot7)(2\cdot0-1\cdot0)}2\cdot846$$ **2**

$= 0\cdot244 - 2\cdot315 + 3\cdot827 + 0\cdot256 = 2\cdot012$ **1**

Not suitable for Newton interpolation as data points are not equally spaced. **1**

A3. Second order relation. **1**

$2a_2 = 4 - 3 + 3 \qquad a_2 = 2$
$2a_3 = 12 - 2 + 3 = 13 \qquad a_3 = 13/2$
$2a_4 = 8 - 13/2 + 3 = 9/2 \qquad a_4 = 9/4$

2

For fixed point, $2a = 4a - a + 3$ so fixed point $a = -3$. **1**

A4. (a) Difference table is:

i	x	$f(x)$	diff1	diff2	diff3
0	2·0	2·317	−185	27	−4
1	2·5	2·132	−158	23	−6
2	3·0	1·974	−135	17	
3	3·5	1·839	−118		
4	4·0	1·721			

2

(b) $p = 0\cdot4$

$$f(3\cdot2) = 1\cdot974 + 0\cdot4(-0\cdot135) + \frac{(0\cdot4)(-0\cdot6)}{2}(0\cdot017)$$ **1**

$$= 1\cdot974 - 0\cdot054 - 0\cdot002 = 1\cdot918$$ **1**

(or, with $p = 2\cdot4$, $2\cdot317 - 0\cdot440 + 0\cdot045$)

A5. In diagonally dominant form, rewritten equations are:

$$x_1 = (5{\cdot}1 + 0{\cdot}2x_2 - 0{\cdot}3x_3)/7$$

$$x_2 = (2{\cdot}5 + 0{\cdot}3x_1)/5$$

$$x_3 = (0{\cdot}4 - 0{\cdot}2x_1 + 0{\cdot}1x_2)/4 \qquad \mathbf{1}$$

This is necessary so that the divisor is larger than the x_i coefficients in the numerator; as a result, convergence can be expected. **1**

Jacobi: $x_1 = 0{\cdot}729$, $x_2 = 0{\cdot}5$; $x_3 = 0{\cdot}1$ **1**

Gauss-Seidel: $x_1 = 0{\cdot}729$; $x_2 = 0{\cdot}544$; $x_3 = 0{\cdot}077$ **1**

Jacobi uses values of x_i from the previous iteration:
Gauss-Seidel uses the most recent values of x_i. **1**

A6. $f(x) = (((0{\cdot}75x + 2{\cdot}16)x)x - 1{\cdot}04)x + 0{\cdot}34$ and $f(1{\cdot}7) = 15{\cdot}448.$ **1,1**

$f(x)_{\max} = (((0{\cdot}755x + 2{\cdot}165)x)x - 1{\cdot}035)x + 0{\cdot}345;$
$f(1{\cdot}7)_{\max} = 15{\cdot}528$ **1**

Maximum absolute error = $15{\cdot}528 - 15{\cdot}448 = 0{\cdot}080$ **1**

Maximum relative error = $\dfrac{0{\cdot}080}{15{\cdot}448} \times 100\% = 0{\cdot}52\%.$ **1**

A7. (a) Taylor expansion gives $f(a) = f(x_0) + f'(x_0)(a - x_0) + \ldots$ and $f(a) = 0.$ **1**

Approximation to a is x_1, so that $f(x_0) + f'(x_0)(a - x_0) = 0.$ **1**

i.e. $x_1 = x_0 - f(x_0)/f'(x_0)$

and in general $x_{n+1} = x_n - f(x_n)/f'(x_n)$ **1**

$f(0) = 1; f(1) = -1{\cdot}09.$ Change of sign means at least one root in interval. **1**

$f'(x) = 2\cos 2x + 3x^2 - 4.$ **1**

$x_0 = 0{\cdot}5;\ x_1 = 0{\cdot}5 - (-0{\cdot}0335)/(-2{\cdot}1694) = 0{\cdot}4846,$ **1**

$x_2 = 0{\cdot}4846.$ Hence root at $0{\cdot}485$ (3D). **1**

(b)

For bisection, $f(1{\cdot}9) = -0{\cdot}353;$ $f(2) = 0{\cdot}243$

$f(1{\cdot}95) = -0{\cdot}073$ **1**

$f(1{\cdot}975) = 0{\cdot}081$

Hence root lies in $[1{\cdot}95, 1{\cdot}975]$. **1**

Using n steps, to reduce interval width by factor of 100 we require $2^n > 100$, $n = 7$ **1**

So 5 more applications are required. **1**

A8. Gaussian elimination table is:

					sum
	(6·3)	2·9	−3·1	8·8	14·9
	1·4	−0·7	2·4	12·2	15·3
	0	(3·7)	1·9	5·1	10·7
$R_2 - 1{\cdot}4R_1/6{\cdot}3$	0	−1·344	3·089	10·244	11·989
$R_4 + 1{\cdot}344R_3/3{\cdot}7$	0	0	3·779	12·097	15·876

4

$x_3 = 3{\cdot}201$; $x_2 = -0{\cdot}265$; $x_1 = 3{\cdot}094$ **1**

To 1D: $x_1 = 3{\cdot}1$ $x_2 = -0{\cdot}3$ $x_3 = 3{\cdot}2$ **1**

There is no reason to expect ill-conditioning since the new elements are comparable in size to the original coefficients. **2**

Guard figures are used to ensure that rounding errors are confined to the guard figures and do not affect the final answers in their rounded form. **1**

A9. Simpson's rule calculation is:

x	$f(x)$	m_4	$m_4 f_1(x)$	m_2	$m_2 f_2(x)$
0	0·7846	1	0·7846	1	0·7846
0·5	0·5317	4	2·1268		
1	0·3478	2	0·6956	4	1·3912
1·5	0·2235	4	0·8940		
2	0·3075	1	0·3075	1	0·3075
			4·8085		2·4833

1

Hence $I_4 = 4{\cdot}8085 \times 0{\cdot}5/3 = 0{\cdot}8014$ and $I_2 = 2{\cdot}4833/3 = 0{\cdot}8278$ **1,1**

Difference table is:

−253	69	−9	157
−184	60	148	
−124	208		
84			

Hence $D = 0{\cdot}157$

2

$|\text{max truncation error}| = 2 \times 0{\cdot}157/180 \approx 0{\cdot}0017$ **1**

Hence $I_4 = 0{\cdot}80$ to suitable accuracy. **1**

With n strips and step size $2h$, the Taylor series for expansion of an integral approximated by Simpson's rule (with principal truncation error of $O(h^4)$ is

$$I = I_n + C(2h)^4 + D(2h)^6 + \ldots = I_n + 16Ch^4 + \ldots \qquad (1)$$

With $2n$ strips and step size h, we have $I = I_{2n} + Ch^4 + Dh^6 + \ldots$ (2)

$16(2) - (1)$ gives $15I = 16I_{2n} - I_n + O(h^6)$

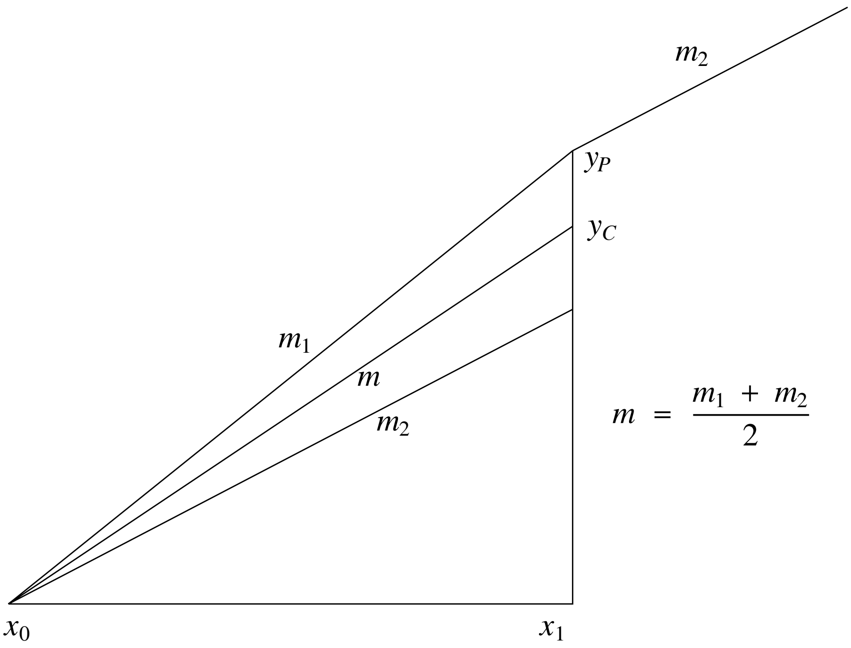
i.e. $I \approx (16I_{2n} - I_n)/15 = I_{2n} + (I_{2n} - I_n)/15$ **3**

Richardson estimate is

$I_R = 0{\cdot}8014 + (0{\cdot}8014 - 0{\cdot}8278)/15 = 0{\cdot}7996$

or 0.800 to suitable accuracy **1**

A10.



y_c is determined using the average of the gradients at $(x_0,\ y_0)$ and $(x_1,\ y_p)$. **3**

Predictor-corrector calculation (with one corrector application) is:

x	y	$y' = e^{-y}(x^2 - 2y)$	y_p	y'_p	$\frac{1}{2}h(y' + y'_p)$
0	1	–0·7358	0·9264	–0·7297	–0·0733
0·1	0·9267	–0·7297	0·8537	–0·7100	–0·0720
0·2	0·8547				

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Section B – Mathematics for Applied Mathematics

B1.

$$\int_0^{\pi/6} x \sin 3x\,dx = \left[x\int \sin 3x\,dx - \int 1.\left(-\frac{1}{3}\cos 3x\right)\right]_0^{\pi/6} \qquad \textbf{2E1}$$

$$= \left[x\left(-\frac{1}{3}\cos 3x\right) + \frac{1}{9}\sin 3x\right]_0^{\pi/6} \qquad \textbf{1}$$

$$= -\frac{\pi}{18}\cos\frac{\pi}{2} + \frac{1}{9}\sin\frac{\pi}{2} - (0 + 0) \qquad \textbf{1}$$

$$= \frac{1}{9} \qquad \textbf{1}$$

B2.

$$\left(x^3 - \frac{2}{x}\right)^4 = \left(x^3\right)^4 + 4\left(x^3\right)^3\left(-\frac{2}{x}\right) + 6\left(x^3\right)^2\left(-\frac{2}{x}\right)^2 + 4x^3\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4 \qquad \textbf{2E1}$$

$$= x^{12} - 8x^8 + 24x^4 - 32 + \frac{16}{x^4} \qquad \textbf{2E1}$$

B3.

$$x = \frac{t}{t^2 + 1} \Rightarrow \frac{dx}{dt} = \frac{1\left(t^2 + 1\right) - t\,(2t)}{\left(t^2 + 1\right)^2} = \frac{1 - t^2}{\left(t^2 + 1\right)^2} \qquad \textbf{2E1}$$

$$y = \frac{t - 1}{t^2 + 1} \Rightarrow \frac{dy}{dt} = \frac{1\left(t^2 + 1\right) - (t - 1)(2t)}{\left(t^2 + 1\right)^2} = \frac{1 + 2t - t^2}{\left(t^2 + 1\right)^2} \qquad \textbf{2E1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1 + 2t - t^2}{(t^2 + 1)^2}}{\frac{1 - t^2}{(t^2 + 1)^2}} \qquad \textbf{1}$$

$$= \frac{1 + 2t - t^2}{1 - t^2}$$

B4.

$$A = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix}$$

$$\det A = \lambda(\lambda - 3) - 4 \qquad \textbf{1}$$

A matrix is singular when its determinant is 0.

$$\lambda^2 - 3\lambda - 4 = 0 \qquad \textbf{1}$$

$$(\lambda + 1)(\lambda - 4) = 0$$

$$\lambda = -1 \text{ or } \lambda = 4 \qquad \textbf{1}$$

When $\lambda = 3$, $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$ so $A^{-1} = \frac{1}{-4}\begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix}$. **1**

B5.

$$x\frac{dy}{dx} - y = x^2e^x$$

$$\frac{dy}{dx} - \frac{1}{x}y = xe^x \qquad \textbf{1}$$

Integrating factor:

$$\exp\left(\int \frac{-1}{x}\,dx\right) \qquad \textbf{1}$$

$$= \exp(-\ln x) = x^{-1} \qquad \textbf{1}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = e^x$$

$$\frac{y}{x} = \int e^x dx = e^x + c \qquad \textbf{1}$$

$$\therefore \quad y = x\left(e^x + c\right)$$

$y = 2$ when $x = 1 \Rightarrow 2 = e + c$

$$\Rightarrow y = x\left(e^x - e + 2\right) \qquad \textbf{1}$$

B6.

$$\frac{8}{x(x+2)(x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4} \qquad \textbf{1}$$

$$8 = A(x+2)(x+4) + Bx(x+4) + Cx(x+2)$$

$$x = 0 \Rightarrow 8A = 8 \Rightarrow A = 1 \qquad \textbf{1}$$

$$x = -2 \Rightarrow -4B = 8 \Rightarrow B = -2 \qquad \textbf{1}$$

$$x = -4 \Rightarrow 8C = 8 \Rightarrow C = 1 \qquad \textbf{1}$$

$$\frac{8}{x(x+2)(x+4)} = \frac{1}{x} + \frac{-2}{x+2} + \frac{1}{x+4}$$

$$\text{Area} = \int_1^2 \left(\frac{1}{x} + \frac{-2}{x+2} + \frac{1}{x+4}\right)dx \qquad \textbf{1M}$$

$$= \left[\ln x - 2\ln(x+2) + \ln(x+4)\right]_1^2 \qquad \textbf{1}$$

$$= \left[\ln\frac{x(x+4)}{(x+2)^2}\right]_1^2$$

$$= \ln\frac{12}{16} - \ln\frac{5}{9} = \ln\frac{12}{16} \times \frac{9}{5} = \ln\frac{27}{20} \qquad \textbf{3E1}$$

[END OF MARKING INSTRUCTIONS]