Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.
   Section A assesses the Units Numerical Analysis 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics
3. **Full credit will be given only where the solution contains appropriate working.**
4. Numerical Analysis Formulae can be found on pages two and three of this Question Paper.
NUMERICAL ANALYSIS FORMULAE

Taylor polynomials

For a function \( f \), defined and \( n \) times differentiable for values of \( x \) close to \( a \), the Taylor polynomial of degree \( n \) is

\[
f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n
\]

and \( f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ldots + \frac{f^{(n)}(a)}{n!}h^n \)

Newton forward difference interpolation formula

\[
f_p = f_0 + \left( \frac{p}{1} \right) \Delta f_0 + \left( \frac{p}{2} \right) \Delta^2 f_0 + \left( \frac{p}{3} \right) \Delta^3 f_0 + \ldots
\]

Lagrange interpolation formula

\[
p_n(x) = \sum_{i=0}^{n} L_i(x) y_i
\]

where \( L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} \)

Newton-Raphson formula

For an equation \( f(x) = 0 \), with \( x_0 \) given,

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

Composite trapezium rule

\[
\int_a^b f(x)dx = \frac{h}{2} \{ f_0 + f_n + 2(f_1 + f_2 + \ldots + f_{n-1}) \} + E
\]

with \( h = \frac{b-a}{n} \) and \( f_k = f(a + kh) \)

where \( |E| \) is (approximately) bounded by

(i) \( \frac{b-a}{12} h^2 M \) with \( |f''(x)| \leq M \) for \( a \leq x \leq b \)

or (ii) \( \frac{b-a}{12} D \) with \( |\Delta^2 f| \leq D \) for \( a \leq x \leq b \)
Simpson’s composite rule

\[
\int_a^b f(x)dx = \frac{h}{3} \left( f_0 + f_{2n} + 4(f_1 + f_3 + \ldots + f_{2n-1}) + 2(f_2 + f_4 + \ldots + f_{2n-2}) \right) + E
\]

with \( h = \frac{b-a}{2n} \) and \( f_k = f(a + kh) \)

where \( |E| \) is (approximately) bounded by

(i) \( \frac{b-a}{180} h^4 M \) with \( |f^{(iv)}(x)| \leq M \) for \( a \leq x \leq b \)

or (ii) \( \frac{b-a}{180} D \) with \( |\Delta^4 f| \leq D \) for \( a \leq x \leq b \)

Richardson’s formula

Trapezium rule: \( I \approx I_{2n} + \frac{1}{3} (I_{2n} - I_n) \)

Simpson’s rule: \( I \approx I_{2n} + \frac{1}{15} (I_{2n} - I_n) \)

Euler’s method

For an equation \( \frac{dy}{dx} = f(x, y) \) with \( (x_0, y_0) \) given,

\( y_{n+1} = y_n + hf(x_n, y_n) \)

Predictor-Corrector method: Euler-Trapezium Rule

\( y_{n+1} = y_n + hf(x_n, y_n) \)

\( y_{n+1} = y_n + \frac{1}{2} h(f(x_n, y_n) + f(x_{n+1}, y_{n+1})) \)

[Turn over]
Section A (Numerical Analysis 1 and 2)

Answer all the questions.

A1. The function $f$ is defined for $|x| < 2.3$ by $f(x) = \ln(7 - 3x)$.

The polynomial $p$ is the Taylor polynomial of degree two for the function $f$ near $x = 2$. Express $p(2 + h)$ in the form $c_0 + c_1 h + c_2 h^2$. Use this polynomial to estimate the value of $f(1.97)$. State, with a reason, whether or not $f(x)$ appears to be sensitive to small changes in $x$ in the neighbourhood of $x = 2$.

A2. The following data are available for a function $f$:

\[
\begin{array}{cccc}
 x & 0.0 & 0.7 & 1.0 & 2.0 \\
 f(x) & 2.716 & 2.315 & 2.103 & 2.846.
\end{array}
\]

Use the cubic Lagrange interpolation formula to estimate $f(1.3)$. Work to three decimal places throughout. Why is it not possible to use the Newton difference formula to estimate $f(1.3)$?

A3. The recurrence relation in $a_r$, with $r = 0, 1, 2, \ldots$, is defined by

\[a_1 = 3 \text{ and } 2a_r = 4a_{r-2} - a_{r-1} + 3, \text{ for } r > 1, \ a_0 = 1.\]

State the order of this recurrence relation. Trace the sequence as far as $a_4$. Determine the fixed point of the recurrence relation.

A4. The following data (accurate to the degree implied) are available for a function $f$:

\[
\begin{array}{ccccccc}
 x & 2.0 & 2.5 & 3.0 & 3.5 & 4.0 \\
\end{array}
\]

(a) Construct a difference table of third order for the data.

(b) Using the Newton forward difference formula of degree two, and working to three decimal places, obtain an approximation to $f(3.2)$. 

A5. Write the following equations in a diagonally dominant form:

\[
\begin{align*}
0 \cdot 2x_1 & - 0 \cdot 1x_2 + 4x_3 = 0 \cdot 4 \\
-0 \cdot 3x_1 & + 5x_2 = 2 \cdot 5 \\
7x_1 & - 0 \cdot 2x_2 + 0 \cdot 3x_3 = 5 \cdot 1.
\end{align*}
\]

Explain why this is necessary when the equations are to be solved using an iterative procedure.

Use the Jacobi iterative procedure, with \( x_1 = x_2 = x_3 = 0 \) as a first approximation, to obtain the first iterates of \( x_1, x_2 \) and \( x_3 \) for the solution of these equations.

Use the Gauss-Seidel iterative procedure, with \( x_1 = x_2 = x_3 = 0 \) as a first approximation, to obtain the first iterates of \( x_1, x_2 \) and \( x_3 \) for the solution of these equations.

Explain why the results differ.

A6. Express the polynomial \( f(x) = 0.75x^4 + 2.16x^3 - 1.04x + 0.34 \) in nested form and evaluate \( f(1.7) \).

Given that the coefficients of \( f(x) \) are rounded to the accuracy implied, determine the maximum possible value of \( f(1.7) \).

Determine the maximum absolute error and maximum relative error in \( f(1.7) \).

A7. (a) The equation \( f(x) = 0 \) has a root at \( x = a \). Use the Taylor series expansion of \( f(x) \) about \( x_0 \), which is known to be close to \( a \), to derive the formula for the Newton-Raphson method of solution of \( f(x) = 0 \).

Show that the equation \( \sin 2x + x^3 - 4x + 1 = 0 \) has a root in the interval \([0, 1]\).

Given that there is only one root in this interval, use the Newton-Raphson method to determine this root correct to three decimal places.

(b) The other positive root of the equation \( \sin 2x + x^3 - 4x + 1 = 0 \) lies in the interval \([1.9, 2]\). Use two applications of the bisection method to determine a more accurate estimate of the interval in which this root lies.

How many further applications of the bisection method would be required in order to reduce the interval width to 0.001?
A8. Use Gaussian elimination with partial pivoting to solve the system of linear equations.

\[
\begin{pmatrix}
0.0 & 3.7 & 1.9 \\
1.4 & -0.7 & 2.4 \\
6.3 & 2.9 & -3.1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
5.1 \\
12.2 \\
8.8
\end{pmatrix}
\]

Incorporate a row check, carry two guard figures in the calculation and record your answers rounded to one decimal place.

State whether you consider this system to be ill-conditioned, giving a reason for your answer.

Explain why guard figures are used in the calculation.

A9. The following data are available for a function \( f \):

\[
\begin{array}{c|c|c|c|c|c}
 x & 0 & 0.5 & 1 & 1.5 & 2 \\
 f(x) & 0.7846 & 0.5317 & 0.3478 & 0.2235 & 0.3075
\end{array}
\]

Use Simpson’s rule with two strips and the composite Simpson’s rule with four strips to obtain two estimates \( I_2 \) and \( I_4 \) respectively for the integral \( I = \int_0^2 f(x) \, dx \). Perform the calculations using four decimal places.

By constructing an appropriate difference table, obtain an estimate of the maximum truncation error in \( I_4 \). Three decimal place accuracy should be used within the table.

Hence state the value of \( I_4 \) to a suitable accuracy.

By considering appropriate Taylor series expansions for a definite integral, establish Richardson’s formula to improve the accuracy of Simpson’s rule by interval halving.

Use Richardson extrapolation to obtain an improved estimate for \( I \) based on the values of \( I_2 \) and \( I_4 \).

A10. A predictor-corrector method for the solution of a differential equation uses Euler’s method as predictor. Explain, with the aid of a diagram, how the trapezium rule may be used as the corrector.

The differential equation

\[
\frac{dy}{dx} = e^{-\gamma}(x^2 - 2y) \text{ with } y(0) = 1
\]

is to be solved numerically. Use the predictor-corrector method with Euler’s method as predictor and the trapezium rule as corrector to obtain a solution of this equation at \( x = 0.2 \). Use one application of the corrector on each step, use a step length \( h = 0.1 \) and perform the calculations using four decimal place accuracy.

[END OF SECTION A]
Section B (Mathematics for Applied Mathematics)

Answer all the questions.

B1. Find the exact value of \( \int_0^{\pi/6} x \sin 3x \, dx \).  

B2. Use the binomial theorem to expand \( \left( x^3 - \frac{2}{x} \right)^4 \) and simplify your answer.

B3. A curve is defined parametrically by \( x = \frac{t}{t^2 + 1}, \ y = \frac{t - 1}{t^2 + 1} \). Obtain \( \frac{dy}{dx} \) as a function of \( t \).

B4. For the matrix \( A = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix} \), find the values of \( \lambda \) such that the matrix is singular. Write down the matrix \( A^{-1} \) when \( \lambda = 3 \).

B5. Obtain the solution of the differential equation
\[
\frac{dy}{dx} - y = x^2 e^x
\]
for which \( y = 2 \) when \( x = 1 \).

B6. Express \( \frac{8}{x(x+2)(x+4)} \) in partial fractions.

Calculate the area under the curve
\[
y = \frac{8}{x^3 + 6x^2 + 8x}
\]
between \( x = 1 \) and \( x = 2 \). Express your answer in the form \( \ln \frac{a}{b} \), where \( a \) and \( b \) are positive integers.

[END OF SECTION B]

[END OF QUESTION PAPER]

[X203/701]