X203/701

NATIONAL QUALIFICATIONS 2007

TUESDAY, 22 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Numerical Analysis

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Numerical Analysis 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics
3. **Full credit will be given only where the solution contains appropriate working.**
4. Numerical Analysis Formulae can be found on pages two and three of this Question Paper.

NUMERICAL ANALYSIS FORMULAE

Taylor polynomials

For a function f, defined and n times differentiable for values of x close to a, the Taylor polynomial of degree n is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ . \ . \ . \ + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and $f(a+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ . \ . \ . \ + \frac{f^{(n)}(a)}{n!}h^n$

Newton forward difference interpolation formula

$$f_p = f_0 + \binom{p}{1}\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \ . \ . \ .$$

Lagrange interpolation formula

$$p_n(x) = \sum_{i=0}^{n} L_i(x) y_i$$

where $L_i(x) = \dfrac{(x-x_0)(x-x_1)\ . \ . \ .\ (x-x_{i-1})(x-x_{i+1})\ . \ . \ .\ (x-x_n)}{(x_i-x_0)(x_i-x_1)\ . \ . \ .\ (x_i-x_{i-1})(x_i-x_{i+1})\ . \ . \ .\ (x_i-x_n)}$

Newton-Raphson formula

For an equation $f(x) = 0$, with x_0 given,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Composite trapezium rule

$$\int_a^b f(x)dx = \frac{h}{2}\{f_0 + f_n + 2(f_1 + f_2 + \ . \ . \ . \ + f_{n-1})\} + E$$

with $h = \dfrac{b-a}{n}$ and $f_k = f(a+kh)$

where $|E|$ is (approximately) bounded by

(i) $\dfrac{b-a}{12}h^2 M$ with $|f''(x)| \le M$ for $a \le x \le b$

or (ii) $\dfrac{b-a}{12}D$ with $|\Delta^2 f| \le D$ for $a \le x \le b$

Simpson's composite rule

$$\int_a^b f(x)dx = \frac{h}{3}\{f_0 + f_{2n} + 4(f_1 + f_3 + \ \ldots \ + f_{2n-1}) + 2(f_2 + f_4 + \ \ldots \ + f_{2n-2})\} + E$$

with $h = \dfrac{b-a}{2n}$ and $f_k = f(a+kh)$

where $|E|$ is (approximately) bounded by

(i) $\dfrac{b-a}{180}h^4 M$ with $\left|f^{(\text{iv})}(x)\right| \le M$ for $a \le x \le b$

or (ii) $\dfrac{b-a}{180}D$ with $\left|\Delta^4 f\right| \le D$ for $a \le x \le b$

Richardson's formula

Trapezium rule: $I \approx I_{2n} + \dfrac{1}{3}(I_{2n} - I_n)$

Simpson's rule: $I \approx I_{2n} + \dfrac{1}{15}(I_{2n} - I_n)$

Euler's method

For an equation $\dfrac{dy}{dx} = f(x, y)$ with (x_0, y_0) given,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Predictor-Corrector method: Euler-Trapezium Rule

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{1}{2}h(f(x_n, y_n) + f(x_{n+1}, y_{n+1}))$$

[Turn over

Section A (Numerical Analysis 1 and 2)

Answer all the questions.

Marks

A1. The function f is defined for $|x| < 2{\cdot}3$ by $f(x) = \ln(7 - 3x)$.

The polynomial p is the Taylor polynomial of degree two for the function f near $x = 2$. Express $p(2 + h)$ in the form $c_0 + c_1h + c_2h^2$. **3**

Use this polynomial to estimate the value of $f(1{\cdot}97)$. **2**

State, with a reason, whether or not $f(x)$ appears to be sensitive to small changes in x in the neighbourhood of $x = 2$. **1**

A2. The following data are available for a function f:

x	0·0	0·7	1·0	2·0
$f(x)$	2·716	2·315	2·103	2·846.

Use the cubic Lagrange interpolation formula to estimate $f(1{\cdot}3)$. Work to three decimal places throughout. **3**

Why is it not possible to use the Newton difference formula to estimate $f(1{\cdot}3)$? **1**

A3. The recurrence relation in a_r, with $r = 0, 1, 2, \ldots$, is defined by

$$a_1 = 3 \text{ and } 2a_r = 4a_{r-2} - a_{r-1} + 3, \text{ for } r > 1,\ a_0 = 1.$$

State the order of this recurrence relation. **1**

Trace the sequence as far as a_4. **2**

Determine the fixed point of the recurrence relation. **1**

A4. The following data (accurate to the degree implied) are available for a function f:

x	2·0	2·5	3·0	3·5	4·0
$f(x)$	2·317	2·132	1·974	1·839	1·721.

(*a*) Construct a difference table of third order for the data. **2**

(*b*) Using the Newton forward difference formula of degree two, and working to three decimal places, obtain an approximation to $f(3{\cdot}2)$. **2**

Marks

A5. Write the following equations in a diagonally dominant form:

$$\begin{aligned} 0\cdot 2x_1 \; - \; 0\cdot 1x_2 \; + \; 4x_3 &= 0\cdot 4 \\ -0\cdot 3x_1 \; + \; 5x_2 \qquad\qquad &= 2\cdot 5 \\ 7x_1 \; - \; 0\cdot 2x_2 \; + \; 0\cdot 3x_3 &= 5\cdot 1. \end{aligned}$$ **1**

Explain why this is necessary when the equations are to be solved using an iterative procedure. **1**

Use the Jacobi iterative procedure, with $x_1 = x_2 = x_3 = 0$ as a first approximation, to obtain the **first iterates** of x_1, x_2 and x_3 for the solution of these equations. **1**

Use the Gauss-Seidel iterative procedure, with $x_1 = x_2 = x_3 = 0$ as a first approximation, to obtain the **first iterates** of x_1, x_2 and x_3 for the solution of these equations. **1**

Explain why the results differ. **1**

A6. Express the polynomial $f(x) = 0{\cdot}75x^4 + 2{\cdot}16x^3 - 1{\cdot}04x + 0{\cdot}34$ in nested form and evaluate $f(1{\cdot}7)$. **2**

Given that the coefficients of $f(x)$ are rounded to the accuracy implied, determine the maximum possible value of $f(1{\cdot}7)$. **1**

Determine the maximum absolute error and maximum relative error in $f(1{\cdot}7)$. **2**

A7. (*a*) The equation $f(x) = 0$ has a root at $x = a$. Use the Taylor series expansion of $f(x)$ about x_0, which is known to be close to a, to derive the formula for the Newton-Raphson method of solution of $f(x) = 0$. **3**

Show that the equation $\sin 2x + x^3 - 4x + 1 = 0$ has a root in the interval [0, 1]. **1**

Given that there is only one root in this interval, use the Newton-Raphson method to determine this root correct to three decimal places. **3**

(*b*) The other positive root of the equation $\sin 2x + x^3 - 4x + 1 = 0$ lies in the interval [1·9, 2]. Use two applications of the bisection method to determine a more accurate estimate of the interval in which this root lies. **2**

How many **further** applications of the bisection method would be required in order to reduce the interval width to 0·001? **2**

[Turn over

Marks

A8. Use Gaussian elimination with partial pivoting to solve the system of linear equations.

$$\begin{pmatrix} 0\cdot0 & 3\cdot7 & 1\cdot9 \\ 1\cdot4 & -0\cdot7 & 2\cdot4 \\ 6\cdot3 & 2\cdot9 & -3\cdot1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5\cdot1 \\ 12\cdot2 \\ 8\cdot8 \end{pmatrix}$$

Incorporate a row check, carry two guard figures in the calculation and record your answers rounded to one decimal place. **6**

State whether you consider this system to be ill-conditioned, giving a reason for your answer. **2**

Explain why guard figures are used in the calculation. **1**

A9. The following data are available for a function f:

x	0	0·5	1	1·5	2
$f(x)$	0·7846	0·5317	0·3478	0·2235	0·3075.

Use Simpson's rule with two strips and the composite Simpson's rule with four strips to obtain two estimates I_2 and I_4 respectively for the integral $I = \int_0^2 f(x)\,dx$. Perform the calculations using four decimal places. **3**

By constructing an appropriate difference table, obtain an estimate of the maximum truncation error in I_4. Three decimal place accuracy should be used within the table. **3**

Hence state the value of I_4 to a suitable accuracy. **1**

By considering appropriate Taylor series expansions for a definite integral, establish Richardson's formula to improve the accuracy of Simpson's rule by interval halving. **3**

Use Richardson extrapolation to obtain an improved estimate for I based on the values of I_2 and I_4. **1**

A10. A predictor-corrector method for the solution of a differential equation uses Euler's method as predictor. Explain, with the aid of a diagram, how the trapezium rule may be used as the corrector. **3**

The differential equation

$$\frac{dy}{dx} = e^{-y}(x^2 - 2y) \text{ with } y(0) = 1$$

is to be solved numerically. Use the predictor-corrector method with Euler's method as predictor and the trapezium rule as corrector to obtain a solution of this equation at $x = 0\cdot2$. Use **one** application of the corrector on each step, use a step length $h = 0\cdot1$ and perform the calculations using four decimal place accuracy. **6**

[*END OF SECTION A*]

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

B1. Find the exact value of $\int_0^{\pi/6} x \sin 3x \, dx$. **5**

B2. Use the binomial theorem to expand $\left(x^3 - \frac{2}{x}\right)^4$ and simplify your answer. **4**

B3. A curve is defined parametrically by $x = \frac{t}{t^2+1}$, $y = \frac{t-1}{t^2+1}$.

Obtain $\frac{dy}{dx}$ as a function of t. **5**

B4. For the matrix $A = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix}$, find the values of λ such that the matrix is singular. **3**

Write down the matrix A^{-1} when $\lambda = 3$. **1**

B5. Obtain the solution of the differential equation

$$x\frac{dy}{dx} - y = x^2e^x$$

for which $y = 2$ when $x = 1$. **5**

B6. Express $\frac{8}{x(x+2)(x+4)}$ in partial fractions. **4**

Calculate the area under the curve

$$y = \frac{8}{x^3 + 6x^2 + 8x}$$

between $x = 1$ and $x = 2$. Express your answer in the form $\ln\frac{a}{b}$, where a and b are positive integers. **5**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]

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