2006 Applied Mathematics

Advanced Higher – Numerical Analysis

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, is accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated, full credit will be given for an alternative valid method.

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question 1, 1M, 1, 1 means a method mark for the product rule (and then a mark for each of the terms). E is shorthand for error. So for example, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Applied Mathematics 2006
Section A – Numerical Analysis

A1. $$L(x) = \frac{(x-3)(x-4)}{(1-3)(1-4)}(-0{\cdot}324) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(0{\cdot}683) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(0{\cdot}914)$$
$$= \frac{(x^2-7x+12)}{6}(-0{\cdot}324) - \frac{(x^2-5x+4)}{2}(0{\cdot}683) + \frac{(x^2-4x+3)}{3}(0{\cdot}914)$$
$$= -0{\cdot}091x^2 + 0{\cdot}867x - 1{\cdot}100.$$ **4**

A2. $f(x) = \cos x \quad f'(x) = -\sin x \quad f''(x) = -\cos x \quad f'''(x) = \sin x$

Taylor polynomial is:
$$p\left(\frac{\pi}{3} + h\right) = \cos\frac{\pi}{3} - \sin\frac{\pi}{3}h - \cos\frac{\pi}{3}\frac{h^2}{2} + \sin\frac{\pi}{3}\frac{h^3}{6}$$
$$= 0{\cdot}5 - 0{\cdot}8660h - 0{\cdot}25h^2 + 0{\cdot}1443h^3.$$ **2**

For $\cos 62°$, $h = \pi/90$;
$$p(\pi/3 + \pi/90) = 0{\cdot}5 - 0{\cdot}03023 - 0{\cdot}00030 = 0{\cdot}4695.$$ **2**

Principal truncation error term is $\left(\frac{\sqrt{3}}{12}\right)\left(\frac{\pi}{90}\right)^3 = 0{\cdot}00001.$

Hence a suitable accuracy for second degree approximation is 0·4695. **2**

A3. Let quadratic through $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$ be
$y = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1)$. Then
$$f_0 = A_0; \qquad f_1 = A_0 + A_1h; \qquad f_2 = A_0 + 2A_1h + 2A_2h^2$$
and so
$$A_1 = \frac{f_1 - f_0}{h} = \frac{\Delta f_0}{h}; \qquad A_2 = \frac{f_2 - 2f_1 + f_0}{2h^2} = \frac{\Delta^2 f_0}{2h^2}.$$
Thus
$$y = f_0 + \frac{x - x_0}{h}\Delta f_0 + \frac{(x - x_0)(x - x_1)}{2h^2}\Delta^2 f_0.$$
Setting $x = x_0 + ph$, where $0 < p < 1$, gives
$$y = f_0 + p\Delta f_0 + \tfrac{1}{2}p(p - 1)\Delta^2 f_0.$$ **5**

(Can also be done by an operator expansion of $(1 + \Delta)^p$.)

A4. (a) Maximum rounding error in $\Delta^3 f_0$ is $2^3 \times 0{\cdot}00005 = 0{\cdot}0004$. **1**

(b) $\Delta^2 f_1 = 0{\cdot}0029$. **1**

(c) Differences are approximately constant within rounding error. **1**

(d) $p = 0{\cdot}8$; $f(3{\cdot}16)$
$$= 1{\cdot}0342 + 0{\cdot}8(0{\cdot}0118) + \frac{(0{\cdot}8)(-0{\cdot}2)}{2}(0{\cdot}0037) + \frac{(0{\cdot}8)(-0{\cdot}2)(-1{\cdot}2)}{6}(-0{\cdot}0008)$$
$$= 1{\cdot}0342 + 0{\cdot}0094 - 0{\cdot}0003 - 0{\cdot}0000 = 1{\cdot}0433.$$ **3**

A5. Jacobi table is:

x_1	x_2	x_3
0	0	0
0·53	0·412	0·749
0·496	0·398	0·738
0·496	0·400	0·739

Hence (2 decimal places) $x_1 = 0{\cdot}50$; $x_2 = 0{\cdot}40$; $x_3 = 0{\cdot}74$. **4**

A6. Synthetic division table is:

1	5	−2	2·3	/ $x - 0{\cdot}8$
	0·8	4·64	2·112	
1	5·8	2·64	4·412	

Hence $Q(x) = x^2 + 5{\cdot}8x + 2{\cdot}64$ and $R = 4{\cdot}412$. **3**

Since $f(x)$ increases on [0.75, 0.85] the largest R occurs when
$x = 0{\cdot}85$.
$R_{\max} = 0{\cdot}85^3 + 5 \times 0{\cdot}85^2 - 2 \times 0{\cdot}85 + 2{\cdot}35 = 4{\cdot}88$ ($R_{\min} = 4{\cdot}08$) **3**

A7. Predictor-corrector calculation (with one corrector application) is:

x	y	$y' = (2x - y^2)e^{-x}$	y_P	y'_P	$\frac{1}{2}h(y' + y'_P)$
1	1	0·3679	1·0368	0·3745	0·0371
1·1	1·0371	0·3743	1·0745	0·3751	0·0375
1·2	1·0746				

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A8. Tableau is:

$$\begin{pmatrix} 4{\cdot}6 & 0 & -3{\cdot}614 & 1{\cdot}170 & -0{\cdot}326 & 0 \\ 0 & 5{\cdot}213 & 1{\cdot}270 & -0{\cdot}522 & 1 & 0 \\ 0 & 0 & 4{\cdot}568 & -0{\cdot}679 & 0{\cdot}009 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4{\cdot}6 & 0 & 0 & 0{\cdot}633 & -0{\cdot}319 & 0{\cdot}791 \\ 0 & 5{\cdot}213 & 0 & -0{\cdot}333 & 0{\cdot}997 & -0{\cdot}278 \\ 0 & 0 & 4{\cdot}568 & -0{\cdot}679 & 0{\cdot}009 & 1 \end{pmatrix} \begin{matrix} (R_1 + 3{\cdot}614R_3/4{\cdot}568) \\ (R_2 - 1{\cdot}270R_3/4{\cdot}568) \\ \end{matrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0{\cdot}138 & -0{\cdot}069 & 0{\cdot}172 \\ 0 & 1 & 0 & -0{\cdot}064 & 0{\cdot}191 & -0{\cdot}053 \\ 0 & 0 & 1 & -0{\cdot}149 & 0{\cdot}002 & 0{\cdot}219 \end{pmatrix} \quad \text{(dividing by diagonal elements)}$$

Hence $A^{-1} = \begin{pmatrix} 0{\cdot}14 & -0{\cdot}07 & 0{\cdot}17 \\ -0{\cdot}06 & 0{\cdot}19 & -0{\cdot}05 \\ -0{\cdot}15 & 0{\cdot}00 & 0{\cdot}22 \end{pmatrix}$. **6**

Ill-conditioning means that a small change in the element(s) of A is likely to cause a large change in the inverse matrix. The strong diagonal dominance of the tableau suggests that ill-conditioning is unlikely here. **3**

Partial pivoting ensures that all row operations involve multiplying by numbers less than 1 so that instabilities are not magnified. **1**

A9. Taylor expansion gives $f(a) = f(x_0) + f'(x_0)(a - x_0) + \ldots$ and $f(a) = 0$.
Approximation to a is x_1, so that $f(x_0) + f'(x_0)(x_1 - x_0) = 0$.

$$\text{i.e.} \qquad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

and in general $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$. **3**

$f(x) = x^3 - 4x + 2$ and $f'(x) = 3x^2 - 4$. $x_0 = -2$.
$x_1 = -2 - 2/8 = -2{\cdot}25$; $x_2 = -2{\cdot}215$; $x_3 = -2{\cdot}214$.
Root is $-2{\cdot}21$ (2 decimal places). **2**

For $g(x) = (x^3 + 2)/4$, $g'(x) = 3x^2/4 << 1$ when $x = 0{\cdot}5$, so probably suitable. **1**

For $x_0 = 0{\cdot}5$, iterates are $x_1 = 0{\cdot}531$; $x_2 = 0{\cdot}537$; $x_3 = 0{\cdot}539$
giving root at 0·54 (2 decimal places). **2**

For bisection, $f(1{\cdot}2) = -1{\cdot}072$; $f(2) = 2$
$f(1{\cdot}6) = -0{\cdot}304$
$f(1{\cdot}8) = 0{\cdot}632$;
$f(1{\cdot}7) = 0{\cdot}113$

Hence root lies in [1·6, 1·7]. **2**

A10. (a) Simpson's rule calculation is:

x	$f(x)$	m_1	$m_1 f(x)$	m_2	$m_2 f(x)$
1	0·6931	1	0·6931	1	0·6931
1·5	2·0617			4	8·2468
2	4·3944	4	17·5776	2	8·7888
2·5	7·8298			4	31·3192
3	12·4766	1	12·4766	1	12·4766
			30·7473		61·5245

Hence $I_2 = 30{\cdot}7473 \times 1/3 = 10{\cdot}2491$
and $I_4 = 61{\cdot}5245 \times 0{\cdot}5/3 = 10{\cdot}2541$

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(b) $f^{(iv)}(1) = 1{\cdot}375$; $f^{(iv)}(3) = 0{\cdot}211$.
Maximum truncation error $\approx 1{\cdot}375 \times 0{\cdot}5^4/180 = 0{\cdot}0010$. **2**
Hence suitable estimate is $I_4 = 10{\cdot}25$. **1**

(c) With n strips and step size $2h$, the Taylor series for expansion of an integral approximated by Simpson's rule (with principal truncation error of $O(h^4)$) is

$$I = I_n + C(2h)^4 + D(2h)^6 + \ldots$$
$$= I_n + 16Ch^4 + \ldots \qquad (1)$$

With $2n$ strips and step size h, we have

$$I = I_{2n} + Ch^4 + Dh^6 + \ldots \qquad (2)$$

$16 \times (2) - (1)$ gives $15I = 16I_{2n} - I_n + O(h^6)$
i.e. $I \approx (16I_{2n} - I_n)/15 = I_{2n} + (I_{2n} - I_n)/15$. **3**

$$I = 10{\cdot}2541 + (10{\cdot}2541 - 10{\cdot}2491)/15 = 10{\cdot}2544$$

or 10·254 to suitable accuracy. **1**

Section B – Mathematics for Applied Mathematics

B1.

$$\left.\begin{array}{ccc}1&1&0\\2&3&1\\2&2&1\end{array}\right|\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array} \rightarrow \left.\begin{array}{ccc}1&1&0\\0&1&1\\0&0&1\end{array}\right|\begin{array}{ccc}1&0&0\\-2&1&0\\-2&0&1\end{array}$$

$$\rightarrow \left.\begin{array}{ccc}1&1&0\\0&1&0\\0&0&1\end{array}\right|\begin{array}{ccc}1&0&0\\0&1&-1\\-2&0&1\end{array}$$

$$\rightarrow \left.\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array}\right|\begin{array}{ccc}1&-1&1\\0&1&-1\\-2&0&1\end{array}$$ **M1, 2E1**

$$\text{So } A^{-1} = \begin{pmatrix}1&-1&1\\0&1&-1\\-2&0&1\end{pmatrix}.$$

Other valid methods of obtaining A^{-1} will be accepted.

$$\begin{array}{ccccccc}x&+&y&&&=&1\\2x&+&3y&+&z&=&2\\2x&+&2y&+&z&=&1\end{array}$$

$$A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = A^{-1}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1&-1&1\\0&1&-1\\-2&0&1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix},$$ **M1,1**

so $x = 0, y = 1, z = -1$.

B2.

$$y = \ln(1 + \sin x)$$

$$\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$$ **M1,1**

$$\text{so } \frac{d^2y}{dx^2} = \frac{(1 + \sin x)(-\sin x) - \cos x \cos x}{(1 + \sin x)^2}$$ **M1,1**

$$= \frac{-\sin x - 1}{(1 + \sin x)^2}$$ **1**

$$= \frac{-1}{(1 + \sin x)}.$$

B3.

$$S_n = \tfrac{1}{6}n(n + 1)(2n + 1)$$ **1**

$$S_{2n+1} = \tfrac{1}{6}(2n + 1)(2n + 2)(4n + 3)$$ **1**

$$2^2 + 4^2 + \ldots + (2n)^2 = 4(1^2 + 2^2 + \ldots + n^2)$$

$$= \tfrac{2}{3}n(n + 1)(2n + 1)$$ **1**

B4.

$$\cos^2 y \frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \int \sec^2 x \; dx \qquad \textbf{M1}$$

$$\text{so} \qquad \ln y = \tan x + c. \qquad \textbf{1,1}$$

When $y = 2, x = 0$ giving $c = \ln 2.$ **1**

Hence $\ln y - \ln 2 = \tan x$, i.e. $\ln \tfrac{1}{2} y = \tan x$

$$\Rightarrow y = 2e^{\tan x}. \qquad \textbf{1}$$

B5. $1 + x^2 = u \Rightarrow x\,dx = \tfrac{1}{2}\,du$ so **1**

$$\int \frac{x^3}{\sqrt{1 + x^2}}\,dx = \int \frac{(u - 1)}{\sqrt{u}} \frac{1}{2}\,du \qquad \textbf{1}$$

$$= \frac{1}{2} \int \left(u^{1/2} - u^{-1/2}\right) du \qquad \textbf{1}$$

$$= \frac{1}{3} u^{3/2} - u^{1/2} + c \qquad \textbf{1}$$

$$= \frac{1}{3}\left(1 + x^2\right)^{3/2} - \left(1 + x^2\right)^{1/2} + c \qquad \textbf{1}$$

$$= \frac{1}{3}\left(x^2 - 2\right)\sqrt{1 + x^2} + c$$

B6. (a)

$$\int_0^1 x\, e^{2x}\,dx = \left[x \int e^{2x}\,dx - \int \tfrac{1}{2}\, e^{2x}\,dx\right]_0^1 \qquad \textbf{M1, 1}$$

$$= \left[\tfrac{1}{2}\, x\, e^{2x} - \tfrac{1}{4}\, e^{2x}\right]_0^1 \qquad \textbf{1}$$

$$= \tfrac{1}{2}\, e^2 - \tfrac{1}{4}\, e^2 + \tfrac{1}{4} = \tfrac{1}{4}\left(e^2 + 1\right) \qquad \textbf{1}$$

(b)

$$\int_0^1 x^2\, e^{2x}\,dx = \left[x^2 \int e^{2x}\,dx\right]_0^1 - \int_0^1 2x.\tfrac{1}{2}\, e^{2x}\,dx \qquad \textbf{1}$$

$$= \left[\tfrac{1}{2} x^2 e^{2x}\right]_0^1 - \int_0^1 x\, e^x\,dx \qquad \textbf{1}$$

$$= \left[\tfrac{1}{2}\, e^2 - 0\right] - \tfrac{1}{4}\left(e^2 + 1\right) = \tfrac{1}{4}\left(e^2 - 1\right) \qquad \textbf{1}$$

(c)

$$\int_0^1 \left(3x^2 + 2x\right) e^{2x}\,dx = 3 \int_0^1 x^2\, e^{2x}\,dx + 2 \int_0^1 x\, e^{2x}\,dx \qquad \textbf{1}$$

$$= \tfrac{3}{4}\left(e^2 - 1\right) + \tfrac{2}{4}\left(e^2 + 1\right) \qquad \textbf{1}$$

$$= \tfrac{1}{4}\left(5e^2 - 1\right)$$

[END OF MARKING INSTRUCTIONS]