X203/701

NATIONAL
QUALIFICATIONS
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MONDAY, 22 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Numerical Analysis

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

    Section A assesses the Units Numerical Analysis 1 and 2
    Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**
4. Numerical Analysis Formulae can be found on pages two and three of this Question Paper.

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NUMERICAL ANALYSIS FORMULAE

Taylor polynomials

For a function f, defined and n times differentiable for values of x close to a, the Taylor polynomial of degree n is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ . \ . \ . \ + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and $f(a+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ . \ . \ . \ + \frac{f^{(n)}(a)}{n!}h^n$

Newton forward difference interpolation formula

$$f_p = f_0 + \binom{p}{1}\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \ . \ . \ .$$

Lagrange interpolation formula

$$p_n(x) = \sum_{i=0}^{n} L_i(x) y_i$$

where $L_i(x) = \dfrac{(x-x_0)(x-x_1)\ . \ . \ .\ (x-x_{i-1})(x-x_{i+1})\ . \ . \ .\ (x-x_n)}{(x_i-x_0)(x_i-x_1)\ . \ . \ .\ (x_i-x_{i-1})(x_i-x_{i+1})\ . \ . \ .\ (x_i-x_n)}$

Newton-Raphson formula

For an equation $f(x) = 0$, with x_0 given,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Composite trapezium rule

$$\int_a^b f(x)dx = \frac{h}{2}\{f_0 + f_n + 2(f_1 + f_2 + \ . \ . \ . \ + f_{n-1})\} + E$$

with $h = \dfrac{b-a}{n}$ and $f_k = f(a+kh)$

where $|E|$ is (approximately) bounded by

(i) $\dfrac{b-a}{12}h^2 M$ with $|f''(x)| \le M$ for $a \le x \le b$

or (ii) $\dfrac{b-a}{12}D$ with $|\Delta^2 f| \le D$ for $a \le x \le b$

Simpson's composite rule

$$\int_a^b f(x)dx = \frac{h}{3}\{f_0 + f_{2n} + 4(f_1 + f_3 + \ ... \ + f_{2n-1}) + 2(f_2 + f_4 + \ ... \ + f_{2n-2})\} + E$$

with $h = \dfrac{b-a}{2n}$ and $f_k = f(a+kh)$

where $|E|$ is (approximately) bounded by

(i) $\dfrac{b-a}{180}h^4 M$ with $\left|f^{(\mathrm{iv})}(x)\right| \leq M$ for $a \leq x \leq b$

or (ii) $\dfrac{b-a}{180}D$ with $\left|\Delta^4 f\right| \leq D$ for $a \leq x \leq b$

Richardson's formula

Trapezium rule: $I \approx I_{2n} + \dfrac{1}{3}(I_{2n} - I_n)$

Simpson's rule: $I \approx I_{2n} + \dfrac{1}{15}(I_{2n} - I_n)$

Euler's method

For an equation $\dfrac{dy}{dx} = f(x, y)$ with (x_0, y_0) given,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Predictor-Corrector method: Euler-Trapezium Rule

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{1}{2}h(f(x_n, y_n) + f(x_{n+1}, y_{n+1}))$$

[Turn over

Section A (Numerical Analysis 1 and 2)

Marks

Answer all the questions.

A1. The following data is available for a function f:

x	1	3	4
$f(x)$	$-0{\cdot}324$	$0{\cdot}683$	$0{\cdot}914$.

Use the Lagrange interpolation formula to obtain a quadratic approximation to $f(x)$, simplifying your answer. **4**

A2. The polynomial p is the Taylor polynomial of degree three for the function f near $x = \pi/3$.

For the function $f(x) = \cos x$, express $p(\pi/3 + h)$ in the form

$$c_0 + c_1h + c_2h^2 + c_3h^3.$$ **2**

Estimate the value of $\cos 62\,°$ using the second degree approximation, stating your answer rounded to four decimal places. **2**

Write down the principal truncation error term for this second degree approximation and calculate its value. Hence state the second degree approximation to an appropriate degree of accuracy. **2**

A3. Derive the Newton forward difference formula of degree two to fit a polynomial through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$. **5**

A4. The following data (accurate to the degree implied) and difference table are available for a function f.

i	x_i	$f(x_i)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	$3{\cdot}0$	$1{\cdot}0342$	118	37	-8	-2
1	$3{\cdot}2$	$1{\cdot}0460$	155	29	-10	1
2	$3{\cdot}4$	$1{\cdot}0615$	184	19	-9	
3	$3{\cdot}6$	$1{\cdot}0799$	203	10		
4	$3{\cdot}8$	$1{\cdot}1002$	213			
5	$4{\cdot}0$	$1{\cdot}1215$				

(*a*) Calculate the maximum rounding error in $\Delta^3 f_0$. **1**

(*b*) Identify the value of $\Delta^2 f_1$ in this table. **1**

(*c*) State a feature of the table which suggests that a third degree polynomial would be a good approximation for this function. **1**

(*d*) Use the Newton forward difference formula of degree three to estimate $f(3{\cdot}16)$, working to four decimal places. **3**

A5. Use the Jacobi iterative procedure with $x_1 = x_2 = x_3 = 0$ as a first approximation to solve the following equations correct to two decimal places.

$$\begin{aligned} 8x_1 + 0\cdot34x_2 + 0\cdot18x_3 &= 4\cdot24 \\ 0\cdot46x_1 + 9x_2 - 0\cdot16x_3 &= 3\cdot71 \\ 0\cdot21x_1 + 10x_3 &= 7\cdot49 \end{aligned}$$ **4**

A6. Use synthetic division to obtain (without rounding) the quotient $Q(x)$ and remainder R when the polynomial $f(x) = x^3 + 5x^2 - 2x + 2\cdot3$ is divided by $g(x) = x - 0\cdot8$. **3**

Given that all coefficients are exact except for the constant terms in $f(x)$ and $g(x)$, which are rounded to the degree of accuracy implied, determine to two decimal places the maximum possible value of R. (You may assume that $f(x)$ is increasing on the interval $[0\cdot7, 0\cdot9]$.) **3**

A7. The differential equation $\frac{dy}{dx} = e^{-x}(2x - y^2)$ with $y(1) = 1$ is to be solved numerically.

Use the predictor-corrector method with Euler's method as predictor and the trapezium rule as corrector to obtain a solution of this equation at $x = 1\cdot2$. Use a step size $h = 0\cdot1$ and **one** application of the corrector on each step. Perform the calculations using four decimal place accuracy. **6**

A8. Gaussian elimination with partial pivoting is used to obtain the inverse of the matrix

$$A = \begin{pmatrix} 4\cdot6 & 1\cdot7 & -3\cdot2 \\ 2\cdot4 & 6\cdot1 & -0\cdot4 \\ 3\cdot1 & 1\cdot1 & 2\cdot4 \end{pmatrix}.$$

During the process, the following augmented matrix is obtained.

$$\begin{pmatrix} 4\cdot6 & 0 & -3\cdot614 & 1\cdot170 & -0\cdot326 & 0 \\ 0 & 5\cdot213 & 1\cdot270 & -0\cdot522 & 1 & 0 \\ 0 & 0 & 4\cdot568 & -0\cdot679 & 0\cdot009 & 1 \end{pmatrix}$$

Complete the determination of the inverse of A, giving all elements in the inverse matrix to two decimal places. **6**

Explain what is meant by an *ill-conditioned* matrix. Is A ill-conditioned? Give a reason for your answer. **3**

Why is the use of partial pivoting important in dealing with an ill-conditioned matrix? **1**

[Turn over

Marks

A9. The equation $f(x) = 0$ has a root close to $x = x_0$. Use the Taylor series expansion of $f(x)$ about $x = x_0$ to derive the Newton-Raphson method of solution of $f(x) = 0$. **3**

It is given that the equation $x^3 - 4x + 2 = 0$, has three distinct real roots. Use the Newton-Raphson method to determine the root near $x = -2$ correct to two decimal places. **2**

Show that the iterative scheme

$$x_{n+1} = \frac{2 + x_n^3}{4}$$

may be suitable to determine the root near $x = 0{\cdot}5$. **1**

Using $x = 0{\cdot}5$ as a starting value and recording successive iterates to three decimal places, use Simple Iteration to determine this root to two decimal places. **2**

The third root lies in the interval $[1{\cdot}2, 2{\cdot}0]$. Use three applications of the bisection method to determine a more accurate estimate of the interval in which this root lies. **2**

A10. The function f is defined by $f(x) = x^2 \ln(1 + x)$, $x \geq 0$.

(*a*) Use Simpson's rule with two strips and the composite Simpson's rule with four strips to obtain two estimates I_2 and I_4 respectively for the integral $I = \int_1^3 f(x)\,dx$. Perform the calculations using four decimal places. **4**

(*b*) It is given that for this function $f^{(\text{iv})}(x) = -\dfrac{2(x^2 + 4x + 6)}{(x+1)^4}$ and that $f^{(\text{v})}(x)$ has no zero on the interval $[1, 3]$. Use this information to obtain an estimate of the maximum truncation error in I_4. **2**

Hence state the value of I_4 to a suitable accuracy. **1**

(*c*) Establish Richardson's formula to improve the accuracy of Simpson's rule by interval halving. **3**

Use Richardson extrapolation to obtain an improved estimate for I based on the values of I_2 and I_4 obtained in part (*a*) of the question. **1**

[*END OF SECTION A*]

Section B (Mathematics for Applied Mathematics)

Answer all the questions.

Marks

B1. Calculate A^{-1} where $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}$.

Hence solve the system of equations

$$\begin{array}{rcrcrcl} x & + & y & & & = & 1 \\ 2x & + & 3y & + & z & = & 2 \\ 2x & + & 2y & + & z & = & 1. \end{array}$$ **5**

B2. Given that $y = \ln(1 + \sin x)$, where $0 < x < \frac{\pi}{2}$, show that $\frac{d^2y}{dx^2} = \frac{-1}{1+\sin x}$. **5**

B3. Define $S_n = \sum_{r=1}^{n} r^2$, $n \geq 1$. Write down formulae for S_n and S_{2n+1}. **2**

Obtain a formula for $2^2 + 4^2 + \ldots + (2n)^2$. **1**

B4. Solve the differential equation

$$\cos^2 x \frac{dy}{dx} = y,$$

given that $y > 0$ and that $y = 2$ when $x = 0$. **5**

B5. Use the substitution $1 + x^2 = u$ to obtain $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$. **5**

B6. (*a*) Evaluate $\int_0^1 xe^{2x}\,dx$. **4**

(*b*) Use part (*a*) to evaluate $\int_0^1 x^2e^{2x}\,dx$. **3**

(*c*) Hence obtain $\int_0^1 (3x^2 + 2x)e^{2x}\,dx$. **2**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]