Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.
   - Section A assesses the Units Numerical Analysis 1 and 2
   - Section B assesses the Unit Mathematics for Applied Mathematics
3. **Full credit will be given only where the solution contains appropriate working.**
4. Numerical Analysis Formulae can be found on pages two and three of this Question Paper.
NUMERICAL ANALYSIS FORMULAE

Taylor polynomials

For a function $f$, defined and $n$ times differentiable for values of $x$ close to $a$, the Taylor polynomial of degree $n$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and

$$f(a + h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \ldots + \frac{f^{(n)}(a)}{n!}h^n$$

Newton forward difference interpolation formula

$$f_p = f_0 + \left(\frac{p}{1}\right)\Delta f_0 + \left(\frac{p}{2}\right)\Delta^2 f_0 + \left(\frac{p}{3}\right)\Delta^3 f_0 + \ldots$$

Lagrange interpolation formula

$$p_n(x) = \sum_{i=0}^{n} L_i(x)y_i$$

where $L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$

Newton-Raphson formula

For an equation $f(x) = 0$, with $x_0$ given,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Composite trapezium rule

$$\int_{a}^{b} f(x)dx = \frac{h}{2} \{f_0 + f_n + 2(f_1 + f_2 + \ldots + f_{n-1})\} + E$$

with $h = \frac{b-a}{n}$ and $f_k = f(a + kh)$

where $|E|$ is (approximately) bounded by

(i) $\frac{b-a}{12}h^2M$ with $|f''(x)| \leq M$ for $a \leq x \leq b$

or (ii) $\frac{b-a}{12}D$ with $|\Delta^2 f| \leq D$ for $a \leq x \leq b$
Simpson’s composite rule

\[
\int_a^b f(x)dx = \frac{h}{3} (f_0 + f_{2n} + 4(f_1 + f_3 + \ldots + f_{2n-1}) + 2(f_2 + f_4 + \ldots + f_{2n-2})) + E
\]

with \( h = \frac{b-a}{2n} \) and \( f_k = f(a + kh) \)

where \(|E|\) is (approximately) bounded by

(i) \( \frac{b-a}{180} h^4 M \) with \( |f^{(iv)}(x)| \leq M \) for \( a \leq x \leq b \)

or (ii) \( \frac{b-a}{180} D \) with \( |\Delta^4 f| \leq D \) for \( a \leq x \leq b \)

Richardson’s formula

Trapezium rule: \( I \approx I_{2n} + \frac{1}{3} (I_{2n} - I_n) \)

Simpson’s rule: \( I \approx I_{2n} + \frac{1}{15} (I_{2n} - I_n) \)

Euler’s method

For an equation \( \frac{dy}{dx} = f(x, y) \) with \((x_0, y_0)\) given,

\( y_{n+1} = y_n + hf(x_n, y_n) \)

Predictor-Corrector method: Euler-Trapezium Rule

\( y_{n+1} = y_n + hf(x_n, y_n) \)

\( y_{n+1} = y_n + \frac{1}{2} h(f(x_n, y_n) + f(x_{n+1}, y_{n+1})) \)
Section A (Numerical Analysis 1 and 2)

Answer all the questions.

A1. The following data is available for a function $f$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-0.324</td>
<td>0.683</td>
<td>0.914.</td>
</tr>
</tbody>
</table>

Use the Lagrange interpolation formula to obtain a quadratic approximation to $f(x)$, simplifying your answer. 4

A2. The polynomial $p$ is the Taylor polynomial of degree three for the function $f$ near $x = \pi/3$.

For the function $f(x) = \cos x$, express $p(\pi/3 + h)$ in the form

$$c_0 + c_1h + c_2h^2 + c_3h^3.$$ 2

Estimate the value of $\cos 62^\circ$ using the second degree approximation, stating your answer rounded to four decimal places. 2

Write down the principal truncation error term for this second degree approximation and calculate its value. Hence state the second degree approximation to an appropriate degree of accuracy. 2

A3. Derive the Newton forward difference formula of degree two to fit a polynomial through the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$. 5

A4. The following data (accurate to the degree implied) and difference table are available for a function $f$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$f(x_i)$</th>
<th>$\Delta f$</th>
<th>$\Delta^2 f$</th>
<th>$\Delta^3 f$</th>
<th>$\Delta^4 f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3·0</td>
<td>1·0342</td>
<td>118</td>
<td>37</td>
<td>-8</td>
<td>-2</td>
</tr>
<tr>
<td>1</td>
<td>3·2</td>
<td>1·0460</td>
<td>155</td>
<td>29</td>
<td>-10</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3·4</td>
<td>1·0615</td>
<td>184</td>
<td>19</td>
<td>-9</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3·6</td>
<td>1·0799</td>
<td>203</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3·8</td>
<td>1·1002</td>
<td>213</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4·0</td>
<td>1·1215</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Calculate the maximum rounding error in $\Delta^3 f_0$. 1

(b) Identify the value of $\Delta^2 f_1$ in this table. 1

(c) State a feature of the table which suggests that a third degree polynomial would be a good approximation for this function. 1

(d) Use the Newton forward difference formula of degree three to estimate $f(3·16)$, working to four decimal places. 3
A5. Use the Jacobi iterative procedure with \( x_1 = x_2 = x_3 = 0 \) as a first approximation to solve the following equations correct to two decimal places.

\[
\begin{align*}
8x_1 + 0.34x_2 + 0.18x_3 &= 4.24 \\
0.46x_1 + 9x_2 - 0.16x_3 &= 3.71 \\
0.21x_1 + 10x_3 &= 7.49
\end{align*}
\]

A6. Use synthetic division to obtain (without rounding) the quotient \( Q(x) \) and remainder \( R \) when the polynomial \( f(x) = x^3 + 5x^2 - 2x + 2.3 \) is divided by \( g(x) = x - 0.8 \).

Given that all coefficients are exact except for the constant terms in \( f(x) \) and \( g(x) \), which are rounded to the degree of accuracy implied, determine to two decimal places the maximum possible value of \( R \). (You may assume that \( f(x) \) is increasing on the interval \([0.7, 0.9]\).)

A7. The differential equation \( \frac{dy}{dx} = e^{-x} (2x - y^2) \) with \( y(1) = 1 \) is to be solved numerically.

Use the predictor-corrector method with Euler’s method as predictor and the trapezium rule as corrector to obtain a solution of this equation at \( x = 1.2 \). Use a step size \( h = 0.1 \) and one application of the corrector on each step. Perform the calculations using four decimal place accuracy.

A8. Gaussian elimination with partial pivoting is used to obtain the inverse of the matrix

\[
A = \begin{bmatrix}
4.6 & 1.7 & -3.2 \\
2.4 & 6.1 & -0.4 \\
3.1 & 1.1 & 2.4
\end{bmatrix}
\]

During the process, the following augmented matrix is obtained.

\[
\begin{pmatrix}
4.6 & 0 & -3.614 & 1.170 & -0.326 & 0 \\
0 & 5.213 & 1.270 & -0.522 & 1 & 0 \\
0 & 0 & 4.568 & -0.679 & 0.009 & 1
\end{pmatrix}
\]

Complete the determination of the inverse of \( A \), giving all elements in the inverse matrix to two decimal places.

Explain what is meant by an ill-conditioned matrix. Is \( A \) ill-conditioned? Give a reason for your answer.

Why is the use of partial pivoting important in dealing with an ill-conditioned matrix?
A9. The equation \( f(x) = 0 \) has a root close to \( x = x_0 \). Use the Taylor series expansion of \( f(x) \) about \( x = x_0 \) to derive the Newton-Raphson method of solution of \( f(x) = 0 \).

It is given that the equation \( x^3 - 4x + 2 = 0 \), has three distinct real roots. Use the Newton-Raphson method to determine the root near \( x = -2 \) correct to two decimal places.

Show that the iterative scheme

\[
x_{n+1} = \frac{2 + x_n^3}{4}
\]

may be suitable to determine the root near \( x = 0.5 \).

Using \( x = 0.5 \) as a starting value and recording successive iterates to three decimal places, use Simple Iteration to determine this root to two decimal places.

The third root lies in the interval \([1.2, 2.0]\). Use three applications of the bisection method to determine a more accurate estimate of the interval in which this root lies.

A10. The function \( f \) is defined by \( f(x) = x^2 \ln(1 + x) \), \( x \geq 0 \).

(a) Use Simpson's rule with two strips and the composite Simpson's rule with four strips to obtain two estimates \( I_2 \) and \( I_4 \) respectively for the integral \( I = \int_1^3 f(x) \, dx \). Perform the calculations using four decimal places.

(b) It is given that for this function \( f^{(iv)}(x) = -\frac{2(x^3 + 4x + 6)}{(x+1)^4} \) and that \( f^{(v)}(x) \) has no zero on the interval \([1, 3]\). Use this information to obtain an estimate of the maximum truncation error in \( I_4 \).

Hence state the value of \( I_4 \) to a suitable accuracy.

(c) Establish Richardson's formula to improve the accuracy of Simpson's rule by interval halving.

Use Richardson extrapolation to obtain an improved estimate for \( I \) based on the values of \( I_2 \) and \( I_4 \) obtained in part (a) of the question.
Section B (Mathematics for Applied Mathematics)

Answer all the questions.

B1. Calculate $A^{-1}$ where $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}$.

Hence solve the system of equations

\begin{align*}
    x + y &= 1 \\
    2x + 3y + z &= 2 \\
    2x + 2y + z &= 1.
\end{align*}

B2. Given that $y = \ln(1 + \sin x)$, where $0 < x < \frac{\pi}{2}$, show that $\frac{d^2y}{dx^2} = \frac{-1}{1+\sin x}$.

B3. Define $S_n = \sum_{r=1}^{n} r^2$, $n \geq 1$. Write down formulae for $S_n$ and $S_{2n+1}$.

Obtain a formula for $2^2 + 4^2 + \ldots + (2n)^2$.

B4. Solve the differential equation $\cos^2 x \frac{dy}{dx} = y$,

given that $y > 0$ and that $y = 2$ when $x = 0$.

B5. Use the substitution $1 + x^2 = u$ to obtain $\int \frac{x^3}{\sqrt{1+x^2}} \, dx$.

B6. (a) Evaluate $\int_{0}^{1} xe^{2x} \, dx$.

(b) Use part (a) to evaluate $\int_{0}^{1} x^2 e^{2x} \, dx$.

(c) Hence obtain $\int_{0}^{1} (3x^2 + 2x)e^{2x} \, dx$.

[END OF SECTION B]

[END OF QUESTION PAPER]