SCOTTISH
QUALIFICATIONS
AUTHORITY

2005 Applied Mathematics

Advanced Higher – Numerical Analysis

Finalised Marking Instructions

These Marking Instructions have been prepared by Examination Teams for use by SQA Appointed Markers when marking External Course Assessments.

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, is accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated, full credit will be given for an alternative valid method.

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question 1, 1M, 1, 1 means a method mark for the product rule (and then a mark for each of the terms). E is shorthand for error. In question 3, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Section A – Numerical Analysis

A1. $\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0$ **1**

$\Delta^3 f_0 = (f_3 - 2f_2 + f_1) - (f_2 - 2f_1 + f_0) = f_3 - 3f_2 + 3f_1 - f_0$ **1**

Maximum error is $\varepsilon + 3\varepsilon + 3\varepsilon + \varepsilon = 8\varepsilon$. **1**

A2. (a) $\Delta^2 f_2 = 0{\cdot}020$ **1**

(b) Values of $\Delta^2 f$ are constant within rounding error and values of $\Delta^3 f$ are effectively zero within rounding error. **2**

(c) $p = 0{\cdot}3$

$f(0{\cdot}73) = 1{\cdot}513 + 0{\cdot}3\,(0{\cdot}246) + \dfrac{(0{\cdot}3)\,(-0{\cdot}7)}{2}\,(0{\cdot}02)$ **1**

$= 1{\cdot}513 + 0{\cdot}074 - 0{\cdot}002 = 1{\cdot}585$ **1**

A3. $L(x) = \dfrac{(x-2)(x-4)}{(-1)(-3)}0{\cdot}717 + \dfrac{(x-1)(x-4)}{(1)(-2)}1{\cdot}342 + \dfrac{(x-1)(x-2)}{(2)(3)}0{\cdot}972$ **1**

$= \dfrac{1}{3}\left(x^2 - 6x + 8\right)0{\cdot}717 - \dfrac{1}{2}\left(x^2 - 5x + 4\right)1{\cdot}342 + \dfrac{1}{6}\left(x^2 - 3x + 2\right)0{\cdot}972$ **1**

$= -0{\cdot}270x^2 + 1{\cdot}435x - 0{\cdot}448$ **1**

A4. $T_2(h) = \cos x = \cos(\pi/4 + h)$

$= \cos\pi/4 - h\sin\pi/4 - (h^2/2)\cos\pi/4$ **1**

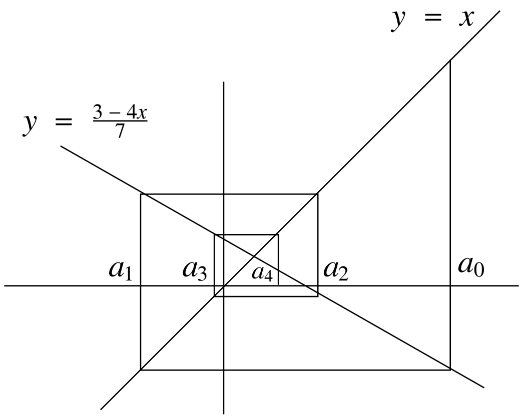
$= \dfrac{\left(1 - h - \frac{1}{2}h^2\right)}{\sqrt{2}}$ **1**

$\cos 46° = \cos\left(\dfrac{\pi}{4} + \dfrac{\pi}{180}\right) \approx \dfrac{1}{\sqrt{2}}\left(1 - \dfrac{\pi}{180}\right) \approx 0{\cdot}7071 - 0{\cdot}0123 = 0{\cdot}695$ (3D) **2**

Three decimal place accuracy is appropriate as second degree term $\approx$ 0·0001 **1**

A5. $a_{n+1} = \left(3 - 4a_n\right)/7,\ a_0 = 2{\cdot}5$

$a_1 = -1;\ a_2 = 1;\ a_3 = -0{\cdot}14;\ a_4 = 0{\cdot}51$ **2**



3

Fixed point has $7a + 4a = 3$, i.e. $a = \frac{3}{11} \sim 0{\cdot}29$ **1**

A6. Division table is:

1	2	−5	2·3	| $x - 0{\cdot}7$
1	2·7	−3·11	0·123	

2

So $Q(x) = x^2 + 2{\cdot}7x - 3{\cdot}11$ and $R = 0{\cdot}123$. **1**

Since $f(x)$ is decreasing, we take $x = 0{\cdot}65$ for maximum: **1**

$R_{\max}(0{\cdot}65) = 0{\cdot}65^3 + 2 \times 0{\cdot}65^2 - 5 \times 0{\cdot}65 + 2{\cdot}35 = 0{\cdot}220$ **1**

A7. Integrate $y' = f(x, y)$ with $x_1 = x_0 + h$

$$\int_{x_0}^{x_1} y'\,dx = \int_{x_0}^{x_1} f(x, y)\,dx$$

Approximating RHS by trapezium rule gives:

$$y_1 - y_0 = f(x_0, y_0)\int_{x_0}^{x_0+h} dx + \text{ terms in } h^2 \text{ and higher powers}$$ **2**

$$y_1 = y_0 + hf(x_0, y_0) + kh^2 + \ldots$$ **1**

and so global truncation error is $kh^2(x_n - x_0)/h = kh(x_n - x_0)$, i.e. first order **1**

Euler calculation for $f(x, y) = \ln(x + y)$ is:

x	y	$f(x, y)$	$hf(x, y)$
1	1	0·6931	0·0693
1·1	1·0693	0·7744	0·0774
1·2	1·1467		

2

A8. (a)

$$\int_{x_0}^{x_1} f(x)\,dx = \int_0^1 f(x_0 + ph)h\,dp = h\int_0^1\left[f(x_0) + f'(x_0)ph + \tfrac{1}{2}f''(x_0)p^2h^2\right]dp$$ **1**

$$= h\left[f(x_0)p + f'(x_0)h\frac{p^2}{2} + f''(x_0)h^2\frac{p^3}{6}\right]_0^1$$

$$= h\left[f(x_0) + f'(x_0)\frac{h}{2} + f''(x_0)\frac{h^2}{6}\right]$$ **2**

$$= h\left[f(x_0) + \tfrac{1}{2}f(x_1) - \tfrac{1}{2}f(x_0) - \tfrac{1}{4}f''(x_0)h^2 + \tfrac{1}{6}f''(x_0)h^2\right]$$

$$= \frac{h(f_0 + f_1)}{2} - \frac{h^3 f''(x_0)}{12}$$ **2**

(using $f(x_1) = f(x_0) + hf'(x_0) + \tfrac{1}{2}h^2 f''(x_0) + \ldots$)

(b) Trapezium rule calculation is:

x	$f(x)$	m_4	$m_4 f(x)$	m_2	$m_2 f(x)$
1	0·36788	1	0·36788	1	0·36788
1·25	0·32032	2	0·64064		
1·5	0·27328	2	0·54656	2	0·54656
1·75	0·22988	2	0·45976		
2	0·19139	1	0·19139	1	0·19139
			2·20623		1·10583

2

Hence $I_2 = 2{\cdot}20623 \times 0{\cdot}25/2 = 0{\cdot}27578$
and $I_1 = 1{\cdot}10583 \times 0{\cdot}5/2 = 0{\cdot}27646$ **1**

Richardson estimate is $I_3 = (4 \times 0{\cdot}27578 - 0{\cdot}27646)/3 = 0{\cdot}2756$ **1**

Trapezium rule estimates are: $I_1 = \frac{2h}{2}(f_0 + 2f_2 + f_4)$

and $I_2 = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4)$ **1**

Applying Richardson gives: $I \approx \frac{4}{3}I_2 - \frac{1}{3}I_1$

$= \frac{4}{3}\frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4) - \frac{h}{3}(f_0 + 2f_2 + f_4)$ **1**

$= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4)$

which is Simpson's Rule with four strips. **1**

A9. (a) Gaussian elimination table is:

					sum
	(4·1)	6·7	−2·9	10·7	18·6
	1·9	3·2	−1·5	5·6	9·2
	0	(2·1)	0·6	7·2	9·9
$R_2 - 1{\cdot}9R_1/4{\cdot}1$	0	0·095	−0·156	0·641	0·580
$R_4 - 0{\cdot}095R_3/2{\cdot}1$	0	0	−0·183	0·315	0·132

4

$x_3 = -1{\cdot}721$; $x_2 = 3{\cdot}920$; $x_1 = -5{\cdot}013$ **1**

To 1D: $x_1 = -5{\cdot}0$; $x_2 = 3{\cdot}9$ $x_3 = -1{\cdot}7$ **1**

(b) Gauss-Seidel table is:

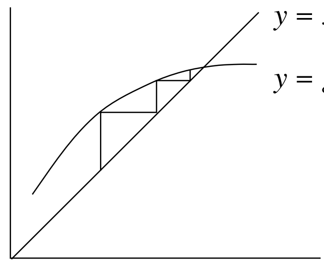
x_1	x_2	x_3
0	0	0
1·431	−0·533	2·635
1·390	−0·450	2·637
1·396	−0·450	2·637

3

Hence (2D) $x_1 = 1{\cdot}40$, $x_2 = -0{\cdot}45$. $x_3 = 2{\cdot}64$ **1**

(c) The (relatively) very small numbers in row 4 of the Gaussian elimination calculation suggest that S_1 is ill-conditioned to some degree. Since S_2 converges rapidly it is clearly not ill-conditioned. **2**

A10. Diagram illustrates $y = x$ and $y = g(x)$ for $0 < g'(x) < 1$ near root.



Iterate x_{n+1} is found by determining intersection of $y = g(x_n)$ and $y = x$.

At each step,
| change in y | < | change in x |
and so $|x_{n+1} - x_n| < |x_n - x_{n-1}|$ giving convergence. **3**

With $g'(x) > 1$ near a, | change in y | > | change in x | gives divergence. **1**

For this iterative scheme $g(x) = (x^3 + 3)/7$ so that $g'(x) = 3x^2/7$. **1**
On [2, 2·5], $g'(x) > 1$ so that scheme is clearly unsuitable.
On [0, 1], $g'(x) < 0{\cdot}5, < 1$ and so the scheme may be suitable. **1**

Using simple iteration, successive iterates are 0·5, 0·446, 0·441, 0·441
Hence root is 0·441, to three decimal places. **2**

For bisection, $f(2) = -3$; $f(2{\cdot}5) = 1{\cdot}125$ **1**
$f(2{\cdot}25) = -1{\cdot}359$
$f(2{\cdot}375) = -0{\cdot}2285$;
$f(2{\cdot}4375) = 0{\cdot}420$ **1**
Hence root lies in [2·375, 2·4375]. **1**

Section B – Mathematics

B1. (a)

$$f(x) = \exp\left(\tan \tfrac{1}{2}x\right)$$

$$f'(x) = \sec^2 \tfrac{1}{2}x \left(\tfrac{1}{2}\right) \exp\left(\tan \tfrac{1}{2}x\right) \qquad \textbf{1,1,1}$$

$$= \tfrac{1}{2} \sec^2 \tfrac{1}{2}x \exp\left(\tan \tfrac{1}{2}x\right)$$

(b)

$$g(x) = \left(x^3 + 1\right) \ln\left(x^3 + 1\right)$$

$$g'(x) = 3x^2 \ln\left(x^3 + 1\right) + \left(x^3 + 1\right) \frac{3x^2}{x^3 + 1} \qquad \textbf{1,1}$$

$$= 3x^2 \ln\left(x^3 + 1\right) + 3x^2 \qquad \textbf{1}$$

B2.

$$A^2 - A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \qquad \textbf{M1}$$

$$= \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \textbf{1}$$

$$= 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \textbf{1}$$

B3.

$$x = 0 \Leftrightarrow 5t^2 - 5 = 0 \Leftrightarrow t = \pm 1 \qquad \textbf{M1}$$

$$y = -3 \Leftrightarrow 3t^3 = -3 \Leftrightarrow t = -1$$

At $(0, -3)$, $t = -1$. **1**

$$\frac{dx}{dt} = 10t; \ \frac{dy}{dt} = 9t^2; \qquad \textbf{1}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9t}{10} \qquad \textbf{1}$$

So when $t = -1$, the gradient is $-\dfrac{9}{10}$. **1**

B4.

$$\left(2a - \frac{3}{a}\right)^4 = (2a)^4 + 4(2a)^3\left(-\frac{3}{a}\right) + 6(2a)^2\left(-\frac{3}{a}\right)^2 + 4(2a)\left(-\frac{3}{a}\right)^3 + \left(-\frac{3}{a}\right)^4 \qquad \textbf{1 powers} \ \ \textbf{1 coeff}$$

$$= 16a^4 - 96a^2 + 216 - \frac{216}{a^2} + \frac{81}{a^4} \qquad \textbf{1}$$

B5.

$$\frac{x^2 + 3}{x\left(1 + x^2\right)} = \frac{A}{x} + \frac{Bx + C}{1 + x^2} \qquad \textbf{M1}$$

$$x^2 + 3 = A\left(1 + x^2\right) + (Bx + C)x$$

$$x = 0 \quad \Rightarrow \quad 3 = A \qquad \textbf{1}$$

$$x^2 + 3 = 3\left(1 + x^2\right) + (Bx + C)x$$

$$x = 1 \quad \Rightarrow \quad 4 = 6 + B + C$$

$$x = -1 \quad \Rightarrow \quad 4 = 6 + B - C$$

$$2C = 0 \Rightarrow C = 0 \text{ and } B = -2 \qquad \textbf{1}$$

$$\int_{1/2}^{1} \frac{x^2 + 3}{x\left(1 + x^2\right)}\, dx = \int_{1/2}^{1} \frac{3}{x} - \frac{2x}{1 + x^2}\, dx \qquad \textbf{1}$$

$$= \left[3 \ln x - \ln\left(1 + x^2\right)\right]_{1/2}^{1} \qquad \textbf{1}$$

$$= \left[0 - \ln 2\right] - \left[3 \ln \tfrac{1}{2} - \ln \tfrac{5}{4}\right]$$

$$= \ln\left(\tfrac{5}{4} \times \tfrac{8}{1} \times \tfrac{1}{2}\right)$$

$$= \ln 5 \approx 1.609 \qquad \textbf{1}$$

B6. (a) *Method 1 − separating the variables*

$$\sin x \frac{dy}{dx} - 2y \cos x = 0$$

$$\frac{dy}{dx} = 2\frac{\cos x}{\sin x} y$$

$$\int \frac{dy}{y} = 2 \int \frac{\cos x}{\sin x} dx \qquad \textbf{M1}$$

$$\ln y = 2 \ln (\sin x) + C \qquad \textbf{1}$$

$$= \ln\left(\sin^2 x\right) + C \qquad \textbf{1}$$

$$y = \exp\left(C + \ln(\sin^2 x)\right) \qquad \textbf{1}$$

$$= e^C \sin^2 x$$

Method 2 − using an integrating factor

$$\sin x \frac{dy}{dx} - 2y \cos x = 0$$

$$\frac{dy}{dx} - 2\frac{\cos x}{\sin x} y = 0 \qquad \textbf{1}$$

$$\int -2\frac{\cos x}{\sin x} dx = -2 \ln (\sin x) = \ln\left(\sin^{-2} x\right) \qquad \textbf{1}$$

Integrating factor = $\exp\left[\ln\left(\sin^{-2} x\right)\right] = \sin^{-2} x$

$$\frac{1}{\sin^2 x}\frac{dy}{dx} + \frac{-2 \cos x}{\sin^3 x} y = 0 \qquad \textbf{1}$$

$$\frac{d}{dx}\left(\frac{y}{\sin^2 x}\right) = 0$$

$$y = A \sin^2 x \qquad \textbf{1}$$

(b)

$$\sin x \frac{dy}{dx} - 2y \cos x = 3 \sin^3 x$$

$$\frac{dy}{dx} - 2\frac{\cos x}{\sin x} y = 3 \sin^2 x. \qquad \textbf{1}$$

Integrating factor is

$$\exp\left(\int -2\frac{\cos x}{\sin x} dx\right) = \exp\left(-2 \ln (\sin x)\right) = \frac{1}{\sin^2 x} \qquad \textbf{M1,1}$$

$$\frac{1}{\sin^2 x}\frac{dy}{dx} + \frac{-2 \cos x}{\sin^3 x} y = 3 \qquad \textbf{1}$$

$$\frac{d}{dx}\left(\frac{y}{\sin^2 x}\right) = 3$$

$$\frac{y}{\sin^2 x} = 3x + D$$

$$y = (3x + D) \sin^2 x \qquad \textbf{1}$$

[END OF MARKING INSTRUCTIONS]