X203/701

NATIONAL QUALIFICATIONS 2005

MONDAY, 23 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Numerical Analysis

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Numerical Analysis 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics
3. **Full credit will be given only where the solution contains appropriate working.**
4. Numerical Analysis Formulae can be found on pages two and three of this Question Paper.

SCOTTISH QUALIFICATIONS AUTHORITY

NUMERICAL ANALYSIS FORMULAE

Taylor polynomials

For a function f, defined and n times differentiable for values of x close to a, the Taylor polynomial of degree n is

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ \dots\ +\frac{f^{(n)}(a)}{n!}(x-a)^n$$

and $f(a+h) \approx f(a)+f'(a)h+\dfrac{f''(a)}{2!}h^2+\ \dots\ +\dfrac{f^{(n)}(a)}{n!}h^n$

Newton forward difference interpolation formula

$$f_p = f_0 + \binom{p}{1}\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \ \dots$$

Lagrange interpolation formula

$$p_n(x) = \sum_{i=0}^{n} L_i(x) y_i$$

where $L_i(x) = \dfrac{(x-x_0)(x-x_1)\ \dots\ (x-x_{i-1})(x-x_{i+1})\ \dots\ (x-x_n)}{(x_i-x_0)(x_i-x_1)\ \dots\ (x_i-x_{i-1})(x_i-x_{i+1})\ \dots\ (x_i-x_n)}$

Newton-Raphson formula

For an equation $f(x)=0$, with x_0 given,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Composite trapezium rule

$$\int_a^b f(x)dx = \frac{h}{2}\{f_0 + f_n + 2(f_1 + f_2 + \ \dots\ + f_{n-1})\} + E$$

with $h = \dfrac{b-a}{n}$ and $f_k = f(a+kh)$

where $|E|$ is (approximately) bounded by

(i) $\dfrac{b-a}{12}h^2 M$ with $|f''(x)| \le M$ for $a \le x \le b$

or (ii) $\dfrac{b-a}{12}D$ with $|\Delta^2 f| \le D$ for $a \le x \le b$

Simpson's composite rule

$$\int_a^b f(x)dx = \frac{h}{3}\{f_0 + f_{2n} + 4(f_1 + f_3 + \ ... \ + f_{2n-1}) + 2(f_2 + f_4 + \ ... \ + f_{2n-2})\} + E$$

with $h = \dfrac{b-a}{2n}$ and $f_k = f(a+kh)$

where $|E|$ is (approximately) bounded by

(i) $\dfrac{b-a}{180}h^4 M$ with $\left|f^{(\text{iv})}(x)\right| \leq M$ for $a \leq x \leq b$

or (ii) $\dfrac{b-a}{180}D$ with $\left|\Delta^4 f\right| \leq D$ for $a \leq x \leq b$

Richardson's formula

Trapezium rule: $I \approx I_{2n} + \dfrac{1}{3}(I_{2n} - I_n)$

Simpson's rule: $I \approx I_{2n} + \dfrac{1}{15}(I_{2n} - I_n)$

Euler's method

For an equation $\dfrac{dy}{dx} = f(x, y)$ with (x_0, y_0) given,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Predictor-Corrector method: Euler-Trapezium Rule

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{1}{2}h(f(x_n, y_n) + f(x_{n+1}, y_{n+1}))$$

[Turn over

Section A (Numerical Analysis 1 and 2)

Marks

Answer all the questions.

A1. In the usual notation for forward differences of function values $f(x)$ tabulated at equally spaced values of x,

$$\Delta f_i = f_{i+1} - f_i$$

where $f_i = f(x_i)$ and $i = \ldots, -2, -1, 0, 1, 2, \ldots$.

Show that $\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$. **2**

If each value of f_i is subject to an error whose magnitude is less than or equal to ε, determine the magnitude of the maximum possible rounding error in $\Delta^3 f_0$. **1**

A2. The following data (accurate to the degree implied) and difference table are available for a function f.

i	x_i	$f(x_i)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	0·5	1·078	207	21	–3
1	0·6	1·285	228	18	2
2	0·7	1·513	246	20	–3
3	0·8	1·759	266	17	
4	0·9	2·025	283		
5	1·0	2·308			

(*a*) Identify the value of $\Delta^2 f_2$ in this table. **1**

(*b*) State a feature of the table which suggests that a polynomial of degree two would be a suitable approximation to f. **2**

(*c*) Using the Newton forward difference formula of degree **two**, and working to three decimal places, estimate $f(0{\cdot}73)$. **2**

A3. The following data are available for a function f.

x	1	2	4
$f(x)$	0·717	1·342	0·972

Use the Lagrange interpolation formula to obtain a quadratic approximation to $f(x)$, simplifying your answer. **3**

A4. Obtain the Taylor polynomial of degree two for the function $\cos x$ near $x = \pi/4$. **2**

Estimate the value of $\cos 46°$ using the first degree approximation. State your answer to the number of figures you would expect to be correct, giving a reason for your choice. **3**

A5. The sequence $\{a_r\}$ is defined by the recurrence relation

$$7a_{n+1} + 4a_n = 3, \qquad \text{with } a_0 = 2{\cdot}5.$$

Trace the sequence as far as the term a_4. (Work to two decimal places where appropriate.) **2**

Display the convergence of the sequence using a cobweb diagram, ie a sketch based on drawing the line segments between the points with coordinates (a_0, a_0), (a_0, a_1), (a_1, a_1), (a_1, a_2), (a_2, a_2), etc. **3**

Determine the fixed point of the sequence. **1**

A6. Use synthetic division to obtain (without rounding) the quotient $Q(x)$ and remainder R when the polynomial $f(x) = x^3 + 2x^2 - 5x + 2{\cdot}3$ is divided by $g(x) = x - 0{\cdot}7$. **3**

It is given that f is decreasing in the neighbourhood of $x = 0{\cdot}7$ and that all coefficients are exact except for the constant terms in $f(x)$ and $g(x)$, which are rounded to the degree of accuracy implied. Determine to three decimal places the maximum possible value of R. **2**

A7. Derive Euler's method for the approximate solution of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

subject to the initial condition $y(x_0) = y_0$ and show that the global truncation error in the process is of first order. **4**

The solution of the differential equation

$$\frac{dy}{dx} = \ln(x + y), \qquad y(1) = 1$$

is required at $x = 1{\cdot}2$. Obtain an approximation to this solution using Euler's method with step size $0{\cdot}1$. Perform the calculation using four decimal place accuracy. **2**

[Turn over

Marks

A8. (*a*) Using a Taylor polynomial of degree two, or otherwise, derive the trapezium rule over a single strip and the corresponding principal error term. **5**

Use the composite trapezium rule with two strips and with four strips to obtain two estimates I_1 and I_2 respectively for the integral $I = \int_1^2 \sqrt{x}\,e^{-x}dx$. Perform the calculations using five decimal places. **3**

Use Richardson extrapolation to obtain an improved estimate I_3 for I based on the values of I_1 and I_2. **1**

(*b*) Values of a function f at five points x_0, x_1, x_2, x_3, x_4, such that $x_{i+1} = x_i + h$ $(0 \leq i \leq 3)$, and spanning an interval, are f_0, f_1, f_2, f_3, f_4. Using Richardson extrapolation on the approximations obtained with the trapezium rule with two and four strips, obtain an estimate of the integral of f on this interval. Show that this result is the same as that obtained using Simpson's rule with four strips. **3**

A9. In the solution of a problem, the following two systems of linear equations, $\mathbf{A}_1\mathbf{x}_1 = \mathbf{B}_1$ and $\mathbf{A}_2\mathbf{x}_2 = \mathbf{B}_2$ have arisen:

$$\text{System } S_1 \ : \quad \mathbf{A}_1 = \begin{pmatrix} 1\cdot 9 & 3\cdot 2 & -1\cdot 5 \\ 4\cdot 1 & 6\cdot 7 & -2\cdot 9 \\ 0\cdot 0 & 2\cdot 1 & 0\cdot 6 \end{pmatrix}, \qquad \mathbf{B}_1 = \begin{pmatrix} 5\cdot 6 \\ 10\cdot 7 \\ 7\cdot 2 \end{pmatrix}$$

$$\text{System } S_2 \ : \quad \mathbf{A}_2 = \begin{pmatrix} 0\cdot 5 & 6\cdot 6 & -0\cdot 2 \\ 5\cdot 1 & -0\cdot 4 & 0\cdot 0 \\ 0\cdot 2 & 0\cdot 0 & 3\cdot 8 \end{pmatrix}, \qquad \mathbf{B}_2 = \begin{pmatrix} -2\cdot 8 \\ 7\cdot 3 \\ 10\cdot 3 \end{pmatrix}$$

(*a*) Use Gaussian elimination with partial pivoting to solve System S_1. Incorporate a row check, show three decimal places in the calculation and round your answer to one decimal place. **6**

(*b*) Use the Gauss-Seidel iterative method with initial values $x_1 = x_2 = x_3 = 0$ to solve System S_2, correct to two decimal places. Show three decimal places in the calculation. **4**

(*c*) State, with a brief explanation, whether there is evidence that either system is ill-conditioned. **2**

Marks

A10. The equation $x = g(x)$ has a root at $x = a$ in an interval I and $g'(x)$ satisfies $0 < g'(x) < 1$ for $x \in I$. An iterative process is defined by $x_{n+1} = g(x_n)$ with a suitable starting value. Explain with the aid of a diagram whether the process will converge when an initial approximation $x = x_0$ within I is chosen. **3**

Explain briefly how the process would differ for the case where $g'(x) > 1$ near $x = a$. **1**

The positive roots of the equation $x^3 - 7x + 3 = 0$ are known to lie within the intervals [0, 1] and [2, 2·5]. For this equation an iterative scheme of the form

$$x_{n+1} = \frac{(x_n^3 + 3)}{7}$$

is proposed for use. Show that this scheme is definitely unsuitable on [2, 2·5], but that it *may* be suitable on [0, 1]. **2**

Using $x = 0{\cdot}5$ as a starting value and recording successive iterates to three decimal places, use Simple Iteration to determine this root to three decimal places. **2**

Use three applications of the bisection method to determine a more accurate estimate of the interval containing the larger positive root. **3**

[*END OF SECTION A*]

[Turn over for Section B on *Page eight*

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

B1. Differentiate, and simplify as appropriate,

(a) $f(x) = \exp(\tan \frac{1}{2}x)$, where $-\pi < x < \pi$, **3**

(b) $g(x) = (x^3 + 1) \ln (x^3 + 1)$, where $x > 0$. **3**

B2. Given that $A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$, show that $A^2 - A = kI$ for a suitable value of k, where I is the 2×2 unit matrix. **3**

B3. A curve is defined by the parametric equations $x = 5t^2 - 5$, $y = 3t^3$.

Find the value of t corresponding to the point $(0, -3)$ and calculate the gradient of the curve at this point. **2, 3**

B4. Expand and simplify $\left(2a - \dfrac{3}{a}\right)^4$. **3**

B5. Express $\dfrac{x^2+3}{x(1+x^2)}$ in partial fractions. **3**

Hence obtain $\displaystyle\int_{1/2}^{1} \frac{x^2+3}{x(1+x^2)}\,dx$. **3**

B6. (a) Given the differential equation

$$\sin x \frac{dy}{dx} - 2y\cos x = 0,$$

find the general solution, expressing y explicitly in terms of x. **4**

(b) Find the general solution of

$$\sin x \frac{dy}{dx} - 2y\cos x = 3\sin^3 x.$$ **5**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]