INTRODUCTION

These support materials for Mathematics were developed as part of the National Qualifications Development Programme in response to needs identified by staff at needs analysis meetings and national seminars.

Advice on learning and teaching may be found in *Achievement for All* (SOEID 1996), *Effective Learning and Teaching in Mathematics* (SOEID 1993), *Improving Mathematics Education 5–14* (SEED 1999) and in the Mathematics Subject Guide.

These notes are intended to support teachers/lecturers in the teaching of Mechanics 2 (Advanced Higher).

Many Mechanics and A-level Mathematics textbooks provide ample work and examples for this unit. Some of these resources are referred to in these notes and are the same as those for Mechanics 1, namely:

- *Understanding Mechanics*
  Sadler and Thorning, Oxford University Press, 019 914675 6
  [abbreviated to S&T]

- *Mechanics*
  R C Solomon, John Murray, 0719570824 [abbreviated to RCS]

- *Mechanics*
  Ted Graham, Collins Educational, 000322372 8 [abbreviated to TG]

- *Mathematics – Mechanics and Probability*
  Bostock and Chandler, Stanley Thornes, 0859501418 [abbreviated to B&C]

- *The Complete A-Level Mathematics*
  Orlando Gough, Heinemann Educational Books, 0435513451
  [abbreviated to OG]

Note
S&T and B&C have no Inverse Square Law of Gravitation and OG has very little. OG has no Simple Harmonic Motion.

SECTION 1

Content

- Know the meaning of the terms angular velocity and angular acceleration

- Know that for motion in a circle of radius r, the radial and tangential components of velocity are $\boldsymbol{0}$ and $r\theta\boldsymbol{e}_\theta$ respectively, and of the acceleration are $-r\dot{\theta}^2\boldsymbol{e}_r$ and $r\ddot{\theta}\boldsymbol{e}_\theta$ respectively, where $\boldsymbol{e}_r = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and $\boldsymbol{e}_\theta = \sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$ are the unit vectors in the radial and tangential directions respectively

- Know the particular case where $\dot{\theta} = \omega t$, ω being constant, when the equations are

  $\boldsymbol{r} = r\cos(\omega t)\boldsymbol{i} + r\sin(\omega t)\boldsymbol{j}$
  $\boldsymbol{v} = -r\omega\sin(\omega t)\boldsymbol{i} + r\omega\cos(\omega t)\boldsymbol{j}$
  $\boldsymbol{a} = r\omega^2\cos(\omega t)\boldsymbol{i} - r\omega^2\sin(\omega t)\boldsymbol{j}$

  from which

  $v = r\omega = r\dot{\theta}$

  $a = r\omega^2 = r\dot{\theta}^2 = \dfrac{v^2}{r}$

  and $\boldsymbol{a} = -\omega^2\boldsymbol{r}$

- Apply these equations to motion in a horizontal circle with uniform angular velocity, including skidding and banking and other applications.

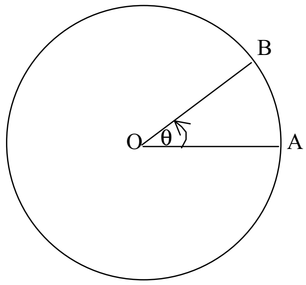
Comments

It is common practice, in writing mathematics, to underline letters representing vectors and in print to denote vectors using bold italic script. Notation can, however, vary from textbook to textbook.

Teaching notes

Angular velocity and angular acceleration

Suppose a particle P moves in a circle centre O and that its initial position is A and after time t it has reached B where angle AOB is θ radians.



The angular speed of P about O, often denoted by ω, is given by $\omega = \dfrac{d\theta}{dt}$ and is measured in radians per second.

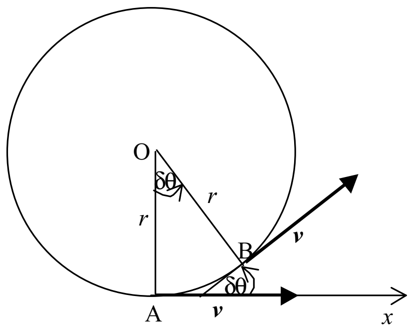
The magnitude of the angular acceleration of P about O is given by $\dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$ and is sometimes denoted by α. It is measured in radians per second per second.

Radial and tangential components of the velocity and acceleration

Some of the textbooks listed do not give the vector development of the equations for circular motion, but use the following technique which students may have met in Physics.

A non-vector approach

Suppose a particle is moving on a circular path of radius r and centre O with constant angular speed ω and that the particle travels from A to B where angle AOB = $\delta\theta$, in time δt.



From the fact that the sum of the angles of a quadrilateral is 2π it follows that the angle between the tangents at A and B will also be $\delta\theta$.

Denoting the arc AB by δs, the linear speed of the particle, v, is given by $v = \dfrac{ds}{dt}$. Now $s = r\theta$ so $v = \dfrac{ds}{dt} = r\dfrac{d\theta}{dt} = r\omega$. Since $v = r\omega$, and ω is constant, the speed of the particle is constant.

As the direction of the motion is along the tangent to the circle

> the tangential component of the velocity is $r\omega$ and the radial component is zero.

Since the direction is constantly changing, the velocity of the particle is not constant and so it is accelerating.

The velocity at B can be resolved into two components, namely
$v \cos\delta\theta$ parallel to Ax
$v \sin\delta\theta$ perpendicular to Ax, i.e. parallel to AO.

The component of acceleration along A*x*

$$= \lim_{\delta t \to 0}\left(\frac{\text{change in velocity component in direction of A}x}{\text{change in time}}\right)$$

$$= \lim_{\delta t \to 0}\left(\frac{v\cos\delta\theta - v}{\delta t}\right) = \lim_{\delta t \to 0}\frac{v(\cos\delta\theta - 1)}{\delta t}$$

$$= \lim_{\delta t \to 0}\frac{v\left(-2\sin^2\frac{\delta\theta}{2}\right)}{\delta t} \qquad \text{[using } \cos 2A = 1 - 2\sin^2 A\text{]}$$

$$= \lim_{\delta t \to 0}\frac{-2v\left(\frac{\delta\theta}{2}\right)^2}{\delta t} \qquad \text{[since } \sin x \to x \text{ as } x \to 0\text{]}$$

$$= \lim_{\delta t \to 0} -\frac{v}{2}\cdot\frac{\delta\theta}{\delta t}\cdot\delta\theta \qquad = -\frac{v}{2}\cdot\omega\cdot 0 = 0$$

The component of acceleration along AO

$$= \lim_{\delta t \to 0}\left(\frac{\text{change in velocity component in direction of AO}}{\text{change in time}}\right)$$

$$= \lim_{\delta t \to 0}\left(\frac{v\sin\delta\theta - 0}{\delta t}\right) = \lim_{\delta t \to 0}\frac{v\sin\delta\theta}{\delta t}$$

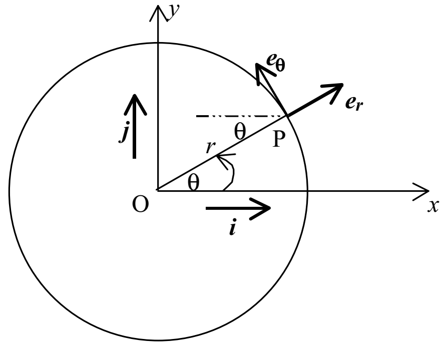
$$= \lim_{\delta t \to 0}\frac{v\delta\theta}{\delta t} = v\omega = r\omega^2 \qquad \text{[since } \sin x \to x \text{ as } x \to 0\text{]}$$

> Thus the tangential component of the acceleration is 0 and the radial component is $r\omega^2\left(=\frac{v^2}{r}\right)$ directed towards the centre of the circle.

Note that the above approach only applies to motion in a circle with constant angular velocity. The following vector approach places no such restriction and considers the situation where the angular velocity is constant as a special case.

A vector approach

Consider a particle P moving round a circle centre O, radius r metres, with the x and y axes as shown. Unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are in the directions of the x and y axes respectively. Unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are directed outwards along the radius and along the tangent respectively as shown. The particle is at position P at time t seconds and OP makes an angle of θ radians with the positive direction of the x-axis.



$\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ can be expressed in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and θ

$\boldsymbol{e}_r = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and $\boldsymbol{e}_\theta = \sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}$

The position vector $\boldsymbol{r}_{\mathrm{P}}$ of P can be written as $\boldsymbol{r}_{\mathrm{P}} = r\boldsymbol{e}_r$, or $\boldsymbol{r}_{\mathrm{P}} = \boldsymbol{r}\cos\theta\boldsymbol{i} + \boldsymbol{r}\sin\theta\boldsymbol{j}$

$$\text{So } \boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -r\sin\theta\frac{d\theta}{dt}\boldsymbol{i} + r\cos\theta\frac{d\theta}{dt}\boldsymbol{j} = r\frac{d\theta}{dt}(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}) \qquad (1)$$

$$= r\frac{d\theta}{dt}\boldsymbol{e}_\theta = r\dot{\theta}\boldsymbol{e}_\theta$$

Thus the velocity has magnitude $r\dfrac{d\theta}{dt} = r\omega$ and is directed along the tangent.

Differentiating (1), using the product rule we obtain

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = r\frac{d^2\theta}{dt^2}[-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}] - r\left(\frac{d\theta}{dt}\right)^2[\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}] \qquad (2)$$

$$= r\frac{d^2\theta}{dt^2}\boldsymbol{e}_\theta - r\left(\frac{d\theta}{dt}\right)^2\boldsymbol{e}_r = r\ddot{\theta}\boldsymbol{e}_\theta - \mathrm{r}\dot{\theta}^2\boldsymbol{e}_r$$

Thus the radial component of the acceleration is $r\dot{\theta}^2$ and is directed towards the centre of the circle and the tangential component is $r\ddot{\theta}$.

Special case $\theta = \omega t$, where ω is constant

$\boldsymbol{r}_P = r\cos(\omega t)\boldsymbol{i} + r\sin(\omega t)\boldsymbol{j}$

Since $\dfrac{d\theta}{dt} = \omega$, (1) above becomes

$\boldsymbol{v} = r\omega[-\sin(\omega t)\boldsymbol{i} + \cos(\omega t)\boldsymbol{j}]$

Since $\dfrac{d\theta}{dt} = \omega$, $\dfrac{d^2\theta}{dt^2} = 0$ and (2) above becomes

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -r\omega^2[\cos(\omega t)\boldsymbol{i} + \sin(\omega t)\boldsymbol{j}] = -\omega^2\boldsymbol{r}_P$$

> Thus, in the special case when $\theta = \omega t$, where ω is constant, the speed of the particle is $v = r\omega = r\dot{\theta}$ and the acceleration is directed towards the centre of the circle with magnitude $r\omega^2 = \dfrac{v^2}{r}$.

Applications to uniform motion in a horizontal circle

If a car is travelling at 15 ms^{-1} round a curve, which is an arc of a circle of radius 100 m, then the acceleration of the car is towards the centre of the circle and its magnitude is '$\dfrac{v^2}{r}$' which equals $\dfrac{15^2}{100} = 2.25\ ms^{-2}$.

Since the Earth moves round the Sun in an orbit which is approximately a circle of radius 1.5×10^8 km, its angular velocity is 2π radians per year (= $\dfrac{2\pi}{365 \times 24 \times 60 \times 60}$ radians per second) and its acceleration towards the Sun is of magnitude '$r\omega^2$' which approximates to

$$1.5 \times 10^{11} \times \frac{4\pi^2}{365^2 \times 24^2 \times 3600^2} = 0.006\ ms^{-2}.$$

Students are aware, from Mechanics 1 (Advanced Higher), that such accelerations necessitate a force in the direction of the acceleration and that '$F = ma$', where m is the mass of the body. Applications to situations of uniform motion in a horizontal circle assume knowledge of forces from Mechanics 1 such as friction (as in the case of the car above), the tension in a string and the normal reaction and the process of resolving forces. The motion of satellites is considered separately later.

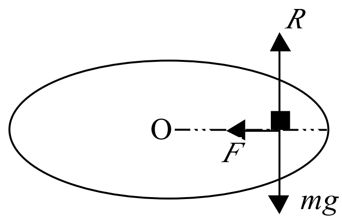
Worked examples

Example 1

A small cube is placed on the surface of a horizontal circular disc, at a distance of 8 cm from the centre of the disc. The cube is on the point of slipping when the disc is rotating at 1.5 revolutions per second. Find the coefficient of friction between the cube and the surface of the disc.

Commentary

The vertical forces acting on the cube are its weight and the normal reaction between the disc and the cube. In the horizontal plane the only force that can be acting on the cube is friction. Since the cube is stationary relative to the disc, it is moving in a horizontal circle with constant angular velocity and so the resultant force acting on the particle must be horizontal and directed along the radius towards the centre of the circle.



Solution

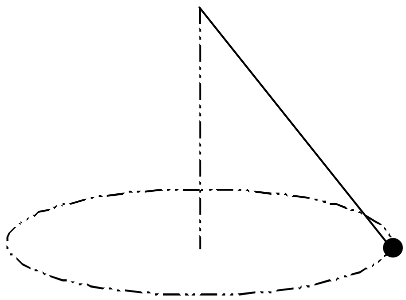
Since there is no vertical motion

– resolving vertically: $R = mg$ (where m is the mass of the cube)
– the frictional force is horizontal of magnitude F.

On the point of slipping $F = \mu R$
Also, '$F = ma$' and '$a = r\omega^2$', hence $F = mr\omega^2$
(where r is the radius of the circle and ω is the angular velocity)

Thus $\mu = \dfrac{F}{R} = \dfrac{mr\omega^2}{mg} = \dfrac{\mathrm{r}\omega^2}{g} = \dfrac{0.08 \times (1.5 \times 2\pi)^2}{9.8} = \dfrac{0.08 \times 9\pi^2}{9.8} = 0.725$

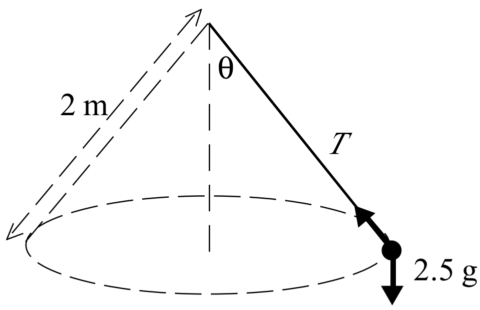
Example 2 (The conical pendulum)



A particle is suspended from one end of a light inextensible string, the other end of which is fixed. Suppose that the particle is made to swing round in horizontal circles below the fixed point. The particle and the string will describe a cone shape in the air, the axis of the cone passing through the fixed point. A system of this type is called a **conical pendulum**.

Suppose the particle has mass 2.5 kg, the string has length 2 m and the angular speed of the particle in the horizontal circle is 3 radians per second. Calculate the tension in the string and the angle it makes with the vertical.

Commentary
The forces acting on the particle are the tension, T, in the string, acting along the string, and the weight of the particle, acting vertically downwards. Since the particle is moving in a horizontal circle with constant angular velocity the resultant force acting on the particle must be horizontal and directed along the radius towards the centre of the circle.



Solution
Since there is no vertical motion

– resolving vertically: $T\cos\theta = 2.5\text{ g}$
(where T newtons is the tension in the string and θ is the angle between the string and the vertical)
– the resultant force is horizontal, resolving horizontally, its magnitude is $T\sin\theta$.

The particle is describing a horizontal circle of radius 2sinθ metres with angular velocity 3 radians per second. Using '$F = ma$' the required horizontal force is '$mr\omega^2$' equal to $2.5 \times 2\sin\theta \times 9$ N.

Hence $\quad T\sin\theta = 2.5 \times 2\sin\theta \times 9 \qquad$ and so $T = 45$

Also $\quad \cos\theta = \dfrac{2.5 \times 9.8}{45} \qquad$ and so $\theta = 57°$

Thus the tension in the string is 45 newtons and the string makes an angle of 57° with the vertical.

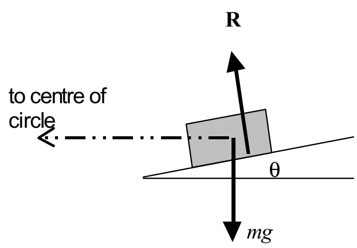
Example 3

A car travels round a bend of radius 500 metres. When its speed is 40 ms^{-1} there is no tendency to slip. Find the angle of banking.

Commentary

As the car is travelling with uniform speed, the driving force is balanced out by resistive forces and consequently the forces acting on the car may be considered to be only its weight and the normal reaction between the road and the car. (Since there is no tendency to slip there is no frictional force.) The car is travelling in a horizontal circle with constant speed and so the resultant force is horizontal and directed towards the centre of this circle.

Cross-section of track



Solution

Since there is no vertical motion

– resolving vertically: $R\cos\theta = mg$ (1)
(where θ is the angle of banking of the road)
– the resultant force is horizontal, resolving horizontally, its magnitude is $R\sin\theta$

But the central force is $\dfrac{mv^2}{r}$. Thus $R\sin\theta = \dfrac{mv^2}{r}$ (2)

From (1) and (2), $\boxed{\tan\theta = \dfrac{v^2}{gr}}$ [A common result which can arise in other situations]

Hence $\tan\theta = \dfrac{1600}{500 \times 9.8}$ and so the angle of banking is 18.1°

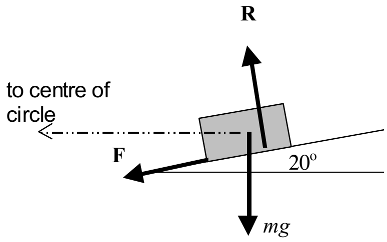
Example 4

A car travels round a bend in a road, which is part of a circle of radius 100 m. The road is banked at 20° to the horizontal. The coefficient of friction between the tyres of the car and the surface of the road is 0.3. Find (i) the greatest speed at which the car can take the bend without slipping outwards and (ii) the least speed at which it can negotiate the bend.

(i) *Commentary*

As for 3 above, we consider the forces acting on the car to be only its weight, the normal reaction of the road on the car and, since the car is about to slip outwards (i.e. up the plane), a frictional force acting downwards parallel to the plane. Since the car is travelling in a horizontal circle with uniform speed the resultant force on the car is horizontal and directed towards the centre of this circle.

Cross-section of track



Solution

Since there is no vertical motion

– resolving vertically: $R\cos 20° = mg + F\sin 20°$

– the resultant force is horizontal, resolving horizontally its magnitude is: $R\sin 20° + F\cos 20°$.

On the point of slipping: $F = \mu R = 0.3R$

Thus: $R\cos 20° - 0.3R\sin 20° = mg$ (1)

Force required to maintain circular motion is: $\dfrac{mv^2}{r} = \dfrac{mv^2}{100}$

(where v is the speed of the car in metres/second)

Thus: $R\sin 20° + 0.3R\cos 20° = \dfrac{mv^2}{100}$ (2)

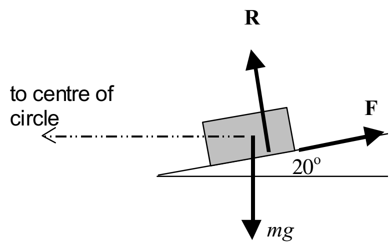
From (1) together with (2) we obtain: $\dfrac{R\left(\sin 20° + 0.3\cos 20°\right)}{R\left(\cos 20° - 0.3\sin 20°\right)} = \dfrac{v^2}{100g}$

Hence $v^2 = \dfrac{980 \times 0.664}{0.891} = 730.5$ and so greatest speed of car is 27 ms^{-1}

(ii) *Commentary*

As for 3 above, we consider the forces acting on the car to be only its weight, the normal reaction of the road on the car and, since the car is about to slip down the plane, a frictional force acting upwards, parallel to the plane. Since the car is travelling in a horizontal circle with uniform speed the resultant force on the car is horizontal and directed towards the centre of this circle.

Cross-section of track



Solution

Since there is no vertical motion

– resolving vertically: $R\cos 20° + F\sin 20° = mg$

– the resultant force is horizontal, resolving horizontally, its magnitude is: $R\sin 20° - F\cos 20°$.

On the point of slipping: $F = \mu R = 0.3R$

Thus: $R\cos 20° + 0.3R\sin 20° = mg$ (1)

Force required to maintain circular motion is: $\dfrac{mv^2}{r} = \dfrac{mv^2}{100}$

(where v is the speed of the car in metres/second)

Thus: $R\sin 20° - 0.3R\cos 20° = \dfrac{mv^2}{100}$ (2)

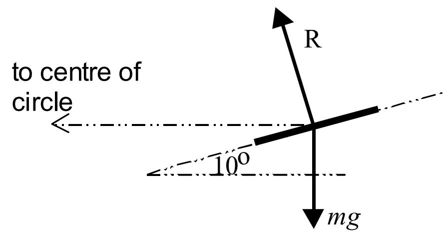
From (1) together with (2) we obtain: $\dfrac{R\left(\sin 20° - 0.3\cos 20°\right)}{R\left(\cos 20° + 0.3\sin 20°\right)} = \dfrac{v^2}{100g}$

Hence $v^2 = \dfrac{980 \times 0.06}{1.04} = 56.4$ and so least speed of car is $7.5\ \text{ms}^{-1}$

Example 5

An aircraft is travelling at 400 ms^{-1} in a horizontal circle. Assuming that the force of the air on the plane is perpendicular to the wings, find the radius of the circle if the angle of banking is 10°.

Commentary



As the aircraft is travelling with uniform speed, the driving force is balanced out by resistive forces and consequently the forces acting on the aircraft may be considered to be only its weight and the force of the air on the wings.
The aircraft is travelling in a horizontal circle with constant speed and so the resultant force is horizontal and directed towards the centre of this circle.

Solution

Since there is no vertical motion

– resolving vertically: $R\cos 10° = mg$ (1)
– the resultant force is horizontal, resolving horizontally, magnitude is $R\sin 10°$.

Magnitude of force required for uniform circular motion is

$\dfrac{mv^2}{r} = \dfrac{160\,000}{r}m$, where r metres is the radius of the circle.

Thus $R\sin 10° = \dfrac{160\,000}{r}m$ (2)

From (1) and (2) $r = \dfrac{160\,000}{9.8 \tan 10°} = 92\,600$

Thus the radius of the circle is 92.6 kilometres.

Note

In Mechanics 1 (Advanced Higher), when dealing with bodies on an inclined plane it was generally helpful to resolve forces parallel and perpendicular to the slope. When horizontal circular motion is involved it is generally more helpful to resolve vertically and horizontally. It is most important that students do not mix the two situations.

Content
- Know Newton's inverse square law of gravitation, namely that the magnitude of the gravitational force of attraction between two particles is inversely proportional to the square of the distance between the two particles
- Apply this to simplified examples of motion of satellites and moons (circular orbits only)
- Find the time for one orbit, height above surface, etc.

Teaching notes

Gravitation

It is only comparatively close to the surface of the Earth that the force of gravity can be considered to be constant.

As a body travels further from the Earth the force of gravity decreases. Newton's inverse square law of gravitation states that the magnitude of the gravitational force of attraction between two particles is inversely proportional to the square of the distance between the two particles. Questions on this topic can be tackled using the principles of variation. Students of Physics will, however, have encountered the more detailed Newton's Universal Law of Gravitation. This law states that, between any two objects in the Universe, there is a force of attraction whose magnitude is given by $F = \frac{Gm_1m_2}{r^2}$, where m_1 and m_2 are the masses of the objects and r is their distance apart. The constant G is given by $G \approx 6.7 \times 10^{-11}$ m^{3}kg^{-1}s^{-2} [Universal constant of gravitation].

However, in Mechanics 2 this result is not required as the relationship $GM = gR^2$ is easy to establish.

Both approaches are illustrated in the first two of the worked examples that follow.

(Incidentally, the mass of the Earth, M, is approximately 5.98×10^{24} kg and the radius of the Earth $\approx 6.4 \times 10^6$ metres. Thus at the surface of the Earth the attraction on a body of mass m kg is given by

$$F = \frac{GMm}{r^2} = \frac{6.7 \times 10^{-11} \times 5.98 \times 10^{24} m}{\left(6.4 \times 10^6\right)^2} \approx 9.8m \text{ N}$$

i.e. at the surface of the Earth $g \approx 9.8$ ms^{-2})

Worked examples

Example 1

A satellite moves with constant speed in a circular orbit around a planet, at the surface of which the magnitude of the acceleration due to gravity is 7.8 ms^{-2}. The radius of the planet is 7500 kilometres and the satellite orbits at a height of h kilometres above the surface of the planet, in a plane through the centre of the planet. At this height the magnitude of the acceleration due to gravity is 6.6 ms^{-2}. Calculate h.

Solution

Newton's inverse square law of gravitation: $a \propto \dfrac{1}{d^2} \Rightarrow a = \dfrac{k}{d^2}$

At the surface of the planet: $7.8 = \dfrac{k}{7\,500\,000^2} \Rightarrow \mathrm{k} = 7.8 \times 7\,500\,000^2$

At height h km above surface: $6.6 = \dfrac{k}{d^2} \Rightarrow d^2 = \dfrac{k}{6.6} = \dfrac{7.8 \times 7\,500\,000^2}{6.6}$

Thus $d = \sqrt{6.648 \times 10^{13}} = 8\,153\,360$
Hence $1000h = 8\,153\,360 - 7\,500\,000 = 653\,360$ and so $h \approx 653$.
Thus satellite is orbiting at a height of 653 kilometres above the Earth's surface.

Alternative solution

Newton's Universal Law of Gravitation: $F = \dfrac{Gm_1m_2}{r^2}$

Thus $mg = \dfrac{GMm}{r^2}$ and hence $gr^2 = GM$

At the surface of the planet: $7.8 \times 7\,500\,000^2 = GM$
At height h km above surface: $6.6d^2 = GM$
Hence $6.6d^2 = 7.8 \times 7\,500\,000^2$

Thus $d^2 = \dfrac{7.8 \times 7\,500\,000^2}{6.6}$, and we continue as before.

Example 2

A satellite moves in a circular orbit around the Earth in the plane of the equator at a height of 920 kilometres above the surface of the Earth, of radius 6400 kilometres. Calculate (a) the magnitude of the acceleration due to gravity at this height and (b) the time for a complete revolution.

Solution

Newton's inverse square law of gravitation: $a \propto \dfrac{1}{d^2} \Rightarrow a = \dfrac{k}{d^2}$

At the surface of the Earth: $9.8 = \dfrac{k}{6\,400\,000^2} \Rightarrow k = 9.8 \times 6\,400\,000^2$

At a height of 920 km above the surface: $g' = \dfrac{9.8 \times 6\,400\,000^2}{7\,320\,000^2} = 7.5 \text{ ms}^{-2}$

Magnitude of force required for uniform circular motion is $mr\omega^2$
Thus $mg' = mr\omega^2$
and so $7.5 = 7\,320\,000\omega^2$

Hence $\omega = \sqrt{\dfrac{7.5}{7\,320\,000}}$

Time for one revolution $= \dfrac{2\pi}{\omega}$ seconds $\approx$ 6207 seconds $\approx$ 1.7 hours

Alternative solution

Newton's Universal Law of Gravitation: $F = \dfrac{Gm_1m_2}{r^2}$

Thus $mg = \dfrac{GMm}{r^2}$ and hence $gr^2 = GM$

At the surface of the Earth: $9.8 \times 6\,400\,000^2 = GM$
At height of 920 km above the Earth $g' \times 7\,320\,000^2 = GM$

Hence $g' = \dfrac{9.8 \times 6\,400\,000^2}{7\,320\,000^2} = 7.5 \text{ ms}^{-2}$, and we continue as before.

Example 3

A satellite moves in a circular orbit around a planet of radius 3500 kilometres. The constant speed of the satellite is 3200 ms^{-1} and it takes 130 minutes to complete one orbit. Calculate the height of the satellite above the surface of the planet. Given that the mass of the satellite is 120 kilograms, and that the magnitude of the acceleration due to gravity at the surface of the planet is 4.2 ms^{-2}, calculate the magnitude of the force due to gravity experienced by the satellite while in orbit.

Solution

$T = \dfrac{2\pi}{\omega}$ so $130 \times 60 = \dfrac{2\pi}{\omega}$

Also $v = r\omega$ so $3200 = d\omega$, where d metres is the radius of the orbit.

Hence, $d = \dfrac{130 \times 60 \times 3200}{2\pi} = 3\,972\,507$

and the height of the satellite above the surface of the planet is 3972.507 – 3500 = 472.5 kilometres

Newton's inverse square law of gravitation: $a \propto \dfrac{1}{d^2} \Rightarrow a = \dfrac{k}{d^2}$

At the surface of the planet: $4.2 = \dfrac{k}{3\,500\,000^2} \Rightarrow \mathrm{k} = 4.2 \times 3\,500\,000^2$

At a height of 472.5 km above surface $g' = \dfrac{4.2 \times 3\,500\,000^2}{3\,972\,507^2}$

Hence, force due to gravity at a height of 472.5 km above surface

$= 120g' = \dfrac{4.2 \times 120 \times 3\,500\,000^2}{3\,972\,507^2} \approx 391$ newtons

Resources/examples

S&T Chapter 13 (Circular Motion)

Pages 305–14	Ex 13A, Pages 307–8
	Ex 13B, Pages 313–5
Pages 315–20	Ex 13C, Pages 320–3 (Conical Pendulum and Banked Tracks)

RCS Chapter 12 (Circular Motion)

Pages 225–9	Ex 12A, Pages 226–7
	Ex 12B, Pages 228–9
Pages 230–5	Ex 12C, Pages 232–3 (Conical Pendulum)
	Ex 12D, Pages 234–5, Nos. 1, 2, 3, 5
Page 237	(Banked Tracks)
Page 236	(Gravitation)

TG Chapter 12 (Circular Motion at Constant Speed)

Pages 226–37	Ex 12.1A, Page 232
	Ex 12.1B, Pages 233–4
Pages 237–41	Ex 12.2A, Pages 238–40 (Banked Tracks) and Ex 12.2B, Pages 240–1 (Gravitation)
Pages 242–51	Ex 12.3A, Pages 245–6 (Conical Pendulum) Nos. 1, 2, 3, 4, 6, 8
	Ex 12.3B, Pages 246–7 (omit Nos. 5 and 7)
Pages 248–9	Consolidation Ex (omit Nos. 3 and 5)

B&C Chapter 10 (Motion in a Circle)

Pages 293–6	Ex 10a, Pages 296–7
Pages 297–9	(Conical Pendulum – omit worked examples 3 and 4)
	Ex 10b, Page 302, Nos. 1–4
Pages 303–7	(Banked Tracks – omit worked example 3)
	Ex 10c, Page 309

OG Chapter 7; 7.4 (Angular Velocity and Circular Motion)

Pages 411–14	Ex 7.4:1, Page 413 (Nos. 1, 4, 14)

Chapter 8; 8.3:5 (Circular Motion)

Pages 465–9	Ex 8.3:5, Pages 467–9 (Conical Pendulum – omit Nos. 11, 12, 22–27, 46)
	Ex 8.1:3, Page 429, Nos. 3–6 (Gravitation)

SECTION 2

Content

- Know the definition of simple harmonic motion (SHM) and the meaning of the terms oscillation, centre of oscillation, period, amplitude, frequency
- Know that SHM can be modelled by the equation $\ddot{x} = -\omega^2 x$
- Know the solutions $x = a\sin(\omega t + \alpha)$ and the special cases $x = a\sin(\omega t)$ and $x = a\cos(\omega t)$ of the SHM equation
- Know and be able to verify that
  $v^2 = \omega^2(a^2 - x^2)$, where $v = \dot{x}$

  $$T = \frac{2\pi}{\omega}$$

  maximum speed is ωa, the magnitude of the maximum acceleration is $\omega^2 a$ and when and where these arise
- Know the meaning of the term tension in the context of elastic strings and springs
- Know Hooke's law, the meaning of the terms natural length, l, modulus of elasticity, λ, and the stiffness constant, k, and the connection between them, $\lambda = kl$
- Know the equation of motion of an oscillating mass and the meaning of the term position of equilibrium
- Apply the above to the solution of problems involving SHM

Comments

At this stage, the solutions of the SHM equations can be verified or established by building on the work of the previous chapter. We can consider the motion of the projection on a diameter, of a particle moving round a circle with constant angular speed. Solutions of second order differential equations are not required.

The result $v^2 = \omega^2(a^2 - x^2)$ is proved here without recourse to solving differential equations since such a proof is not required in this section. A proof involving solving differential equations will arise in the section of work on motion in a straight line later in this unit.

Students will be expected to solve problems on simple harmonic motion involving elastic strings and springs and small amplitude oscillations of a simple pendulum but not the compound pendulum.

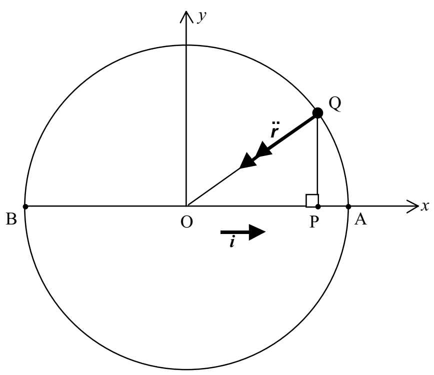
Notation

Please note that some textbooks use 'n' instead of 'ω' in the equation for Simple Harmonic Motion.

Teaching notes

Introduction to simple harmonic motion

Consider a particle, Q, moving round a circle, centre O of radius a, counter-clockwise with constant angular speed ω radians per second.



Let BA be a fixed diameter of the circle and let P be the projection of Q on BA. Taking the origin at O and the x-axis along OA we have

$$x_P = x_Q = a\cos A\hat{O}Q$$

Clearly, as Q moves round the circle, P oscillates between A and B with period $\dfrac{2\pi}{\omega}$.

In the previous chapter we found that the acceleration, $\ddot{\boldsymbol{r}}$, of Q is directed along QO and has magnitude $a\omega^2$.

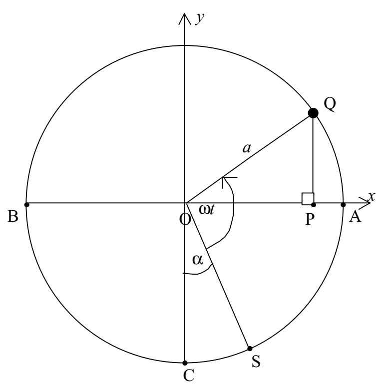
Since the acceleration of P equals the x-component of the acceleration of Q,

$$\ddot{\boldsymbol{x}}_P = -a\omega^2\cos A\hat{O}Q\boldsymbol{i} = -\omega^2(a\cos A\hat{O}Q)\boldsymbol{i} = -\omega^2 x_P\boldsymbol{i} = -\omega^2\boldsymbol{x}_P$$

In other words the acceleration of P is directly proportional to its displacement from O and is directed towards O. Such motion is called simple harmonic motion.

Since the motion is in a straight line, the scalar form of the equation is normally used, namely $\ddot{x} = -\omega^2 x$.

Suppose we time the motion from when the particle passes through S, where $C\hat{O}S = \alpha$ radians as in the diagram on the right. After t seconds the particle will be at Q where $S\hat{O}Q = \omega t$ radians and



$$x_P = a\cos P\hat{O}Q = a\cos(\omega t + \alpha - \frac{\pi}{2})$$

i.e. $x_P = a\sin(\omega t + \alpha)$

Thus the solution of the equation $\ddot{x}_P = -\omega^2 x_P$ corresponding to the particle starting at S, is $\boxed{x_P = a\sin(\omega t + \alpha)}$ [α is called the phase angle].

Alternatively, if the timing starts from:

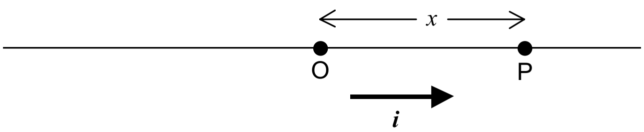
i) when the particle passes through C then $\alpha = 0$ and $x_p = a\sin\omega t$. Thus $\boxed{x_p = a\sin\omega t}$ describes SHM starting at the centre of the oscillation

ii) when the particle passes through A then $\alpha = \dfrac{\pi}{2}$ and $x_p = a\cos\omega t$.

Thus $\boxed{x_p = a\cos\omega t}$ describes SHM starting at the extremity A of the oscillation.

Simple harmonic motion (Starting from the definition)

The motion of a particle which moves in a straight line so that its acceleration is directly proportional to the displacement of the particle from a fixed point on the line, and is always directed towards the fixed point, is called **simple harmonic motion**.



If O is the fixed point on the line, $\boldsymbol{i}$ is the unit vector in the positive direction of the straight line and P is the position of the particle at time t, where OP = x, then the position vector $\boldsymbol{x}$ of P is given by $\boldsymbol{x} = x\boldsymbol{i}$.

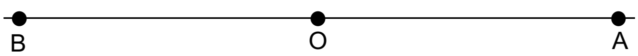
From the definition of SHM the acceleration $\ddot{\boldsymbol{x}}$ at time t is given by the equation

$$\ddot{\boldsymbol{x}} = -\omega^2\boldsymbol{x} = -\omega^2 x\boldsymbol{i}$$

where ω^2 is the constant arising from the direct proportion and the negative sign indicates that, while $\boldsymbol{x}$ is in the direction $\overrightarrow{OP}$, $\ddot{\boldsymbol{x}}$ is in the direction $\overrightarrow{PO}$.

As the motion is in a straight line, the scalar form of the equation is normally used, namely $\boldsymbol{x} = -\omega^2 x$.

Since the acceleration is always in the opposite direction to the displacement, and in the opposite sense, the particle will come to instantaneous rest at some point A. It will then return to O, continue to move in the negative direction until it comes to instantaneous rest again at some point B and continue to oscillate about O between the points A and B.



The distance OA (=OB) is called the **amplitude** of the motion, and it is usually denoted by a.
We can verify, as below, that $\boxed{x = a\sin(\omega t + \alpha)}$ is a solution of the equation $\ddot{x} = -\omega^2 x$, a and α being constants.

$x = a\sin(\omega t + \alpha) \Rightarrow \dot{x} = a\omega\cos(\omega t + \alpha) \Rightarrow \ddot{x} = -a\omega^2\sin(\omega t + \alpha)$
$\Rightarrow \ddot{x} = -\omega^2 x$

Properties of simple harmonic motion

It follows from $x = a\sin(\omega t + \alpha)$ that

- The motion starts at the point where $x = a\sin\alpha$ [α is called the phase angle]
- The maximum value of x is a, confirming that a is the amplitude of the motion
- The motion is periodic and the time for a complete cycle is given by

  $\omega T = 2\pi$, i.e. the period, T, is given by $T = \dfrac{2\pi}{\omega}$

  (It follows that the frequency, $f\left(=\dfrac{1}{T}\right)$, is given by $f = \dfrac{\omega}{2\pi}$)
- The maximum speed is the maximum value of $\dot{x} = a\omega\cos(\omega t + \alpha)$
  which is ωa, occurring when $\cos(\omega t + \alpha) = 1$
  i.e. when $\omega t + \alpha = 2n\pi$, where n is any integer
  i.e. when $x = \sin 2n\pi$
  i.e. when $x = 0$
  i.e. when the particle passes through the centre of the oscillation.

- $\boxed{v^2 = \omega^2(a^2 - x^2)}$
  $v^2 = \dot{x}^2 = a^2\omega^2\cos^2(\omega t + \alpha)$
$\Rightarrow v^2 = a^2\omega^2[1 - \sin^2(\omega t + \alpha)] = \omega^2[a^2 - a^2\sin^2(\omega t + \alpha)] = \omega^2[a^2 - x^2]$

  [As before, we see that the magnitude of the maximum velocity is ωa when $x = 0$]
- The magnitude of the maximum acceleration is the maximum value of $|\ddot{x}|$ which is $\omega^2 a$, occurring when $\sin(\omega t + \alpha) = 1$, i.e. when $x = a$

This latter result can also be seen directly from $\ddot{x} = -\omega^2 x$. $|\ddot{x}|_{MAX} = \omega^2 a$ since a is the magnitude of the maximum displacement of the particle from O in either direction.

Two particular situations are frequently considered.

a) When the motion starts at O
   Here, $x = 0$ when $t = 0$, i.e. $\alpha = 0$, and hence $\boxed{x = a\sin\omega t}$.

b) When the motion starts at the extremity A
   Here, $x = a$ when $t = 0$, i.e. $\alpha = \frac{\pi}{2}$ and hence $\boxed{x = a\cos\omega t}$.

Summarising, for a particle moving with SHM we have

$$\ddot{x} = -\omega^2 x$$
$$x = a\sin(\omega t + \alpha)$$
$x = a\sin(\omega t)$ [Motion starting at centre of oscillation]

$x = a\cos(\omega t)$ [Motion starting at extremity of oscillation]

$$v^2 = \omega^2(a^2 - x^2) \qquad v_{\max} = \omega^2 a \qquad \ddot{x}_{\max} = \omega^2 a$$

$T = \frac{2\pi}{\omega}$ [Sometimes frequency f is used where $f = \frac{1}{T}$]

Hooke's law

When an elastic spring is compressed/extended, or an elastic string extended, the force produced in it is called the tension. It was found experimentally that the tension $\boldsymbol{T}$ produced in the spring (string) is proportional to $\boldsymbol{x}$, the compression/extension of the spring (extension of the string) from its natural length, i.e. $\boldsymbol{T} \propto \boldsymbol{x}$.
Or, in scalar form (since we are dealing with a straight line situation), $T \propto x$

Now $T \propto x \Rightarrow T = kx$, where k is called the stiffness constant of the spring.

This result is **Hooke's law**. If the natural length of the spring (string) is l this is often written as $T = \frac{\lambda x}{l}$, where λ is called the modulus of elasticity of the spring (string) and depends on the material of the spring (string). It follows that $k = \frac{\lambda}{l}$ or $\lambda = kl$.
Incidentally, when $x = l$, $T = \lambda$ and so λ is the magnitude of the force required to double the length of the string (spring).

Motion of a particle suspended by a light string

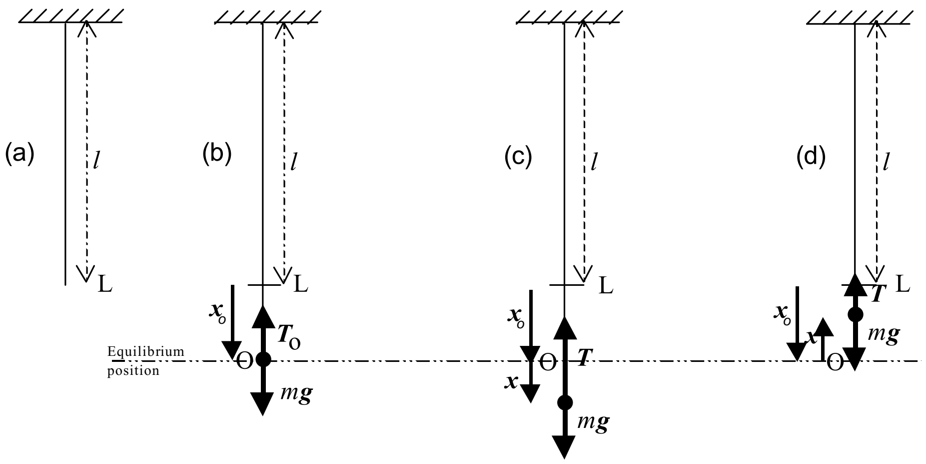
A light, elastic string of natural length l and modulus of elasticity λ hangs vertically attached to a fixed point, as in diagram (a) below.

A particle of mass m is now attached to the end of the string and hangs in **equilibrium** and the tension in the string is $\boldsymbol{T}_o$ as in diagram (b) below.

The particle is now set in motion (either by projecting it downwards/upwards with a given velocity or by pulling it down/up and releasing it).

In diagram (c) the particle is **below** the equilibrium position, the vertical extension is $x_o + x$ and the tension is $\boldsymbol{T}$. (Note that since $x_o + x > x_o$, here $T > T_o$)

In diagram (d) the particle is above the equilibrium position, the vertical extension is $x_o - x$ and the tension is $\boldsymbol{T}$. (Note that since $x_o - x < x_o$, here $T < T_o$)



In diagram (b)

The displacement of the particle from L is $\boldsymbol{x}_o$.

$\boldsymbol{T}_o$ and $\boldsymbol{x}_o$ have opposite sense and so using Hooke's law we have

$$\boldsymbol{T}_o = -\frac{\lambda}{l}\boldsymbol{x}_o .$$

But $\boldsymbol{T}_o + m\boldsymbol{g} = \boldsymbol{0}$, since the system is in equilibrium, hence $\frac{\lambda}{l}x_o = mg$

In both of the diagrams (c) and (d)
The displacement of the particle from L is $\boldsymbol{x}_o + \boldsymbol{x}$.
$\boldsymbol{T}$ and $\boldsymbol{x}_o + \boldsymbol{x}$ have opposite sense and using Hooke's law we have

$\boldsymbol{T} = -\frac{\lambda}{l}(\boldsymbol{x}_o + \boldsymbol{x})$.

The resultant force acting on the particle is $\boldsymbol{T} + m\boldsymbol{g}$ and so the equation of motion is

$$
\begin{aligned}
& m\ddot{\boldsymbol{x}} = \boldsymbol{T} + m\boldsymbol{g} \\
\Rightarrow\ & m\ddot{\boldsymbol{x}} = -\frac{\lambda}{l}(\boldsymbol{x}_o + \boldsymbol{x}) + m\boldsymbol{g} \\
\Rightarrow\ & m\ddot{\boldsymbol{x}} = -\frac{\lambda}{l}\boldsymbol{x}_o - \frac{\lambda}{l}\boldsymbol{x} + m\boldsymbol{g} \\
\Rightarrow\ & m\ddot{\boldsymbol{x}} = -\frac{\lambda}{l}\boldsymbol{x} \quad \text{since} \quad m\boldsymbol{g} = \frac{\lambda}{l}\boldsymbol{x}_o \\
\Rightarrow\ & \ddot{\boldsymbol{x}} = -\frac{\lambda}{ml}\boldsymbol{x}
\end{aligned}
$$

Comparing $\ddot{\boldsymbol{x}} = -\frac{\lambda}{ml}\boldsymbol{x}$ with $\ddot{\boldsymbol{x}} = -\omega^2\boldsymbol{x}$ we see that the particle is moving with simple harmonic motion **about the equilibrium position**, with

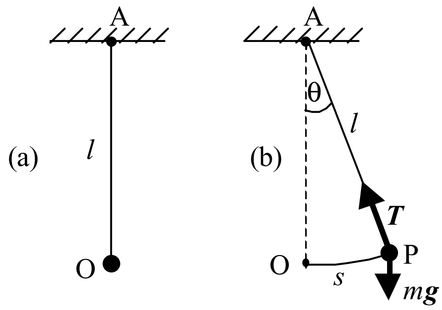
$\omega = \sqrt{\frac{\lambda}{ml}}$.

If the mass is pulled down a metres from the equilibrium position then released, its speed when $x = a$ is zero and so the amplitude of the resulting motion is a metres.

Note: In the work on springs and strings we are assuming that the spring (string) is not over stretched (or over compressed) so that its elasticity is not impaired. With an extending string, as in the last illustration, care must be taken to ensure that the particle does not oscillate above the position of the natural length of the string. If it did so, then there would be no tension in the string and the particle would be moving freely under the constant force of gravity until the string became taut again.

The simple pendulum

Consider a particle of mass m kilograms suspended from a fixed point A by a light **inextensible** string of length l metres. Initially the particle is at a point O vertically below A as shown in diagram (a).



It is then pulled aside, keeping the string taut, so that the string makes a small angle with the vertical, and then released.

Diagram (b) shows the particle at P, where $O\hat{A}P = \theta$ radians, and the arc OP is s metres.

In diagram (b)

$$s = l\theta$$

$$\Rightarrow \dot{s} = l\dot{\theta} \text{ (since } l \text{ is constant)}$$

$$\Rightarrow \ddot{s} = l\ddot{\theta}$$

The forces acting on the particle are the tension in the string $\boldsymbol{T}$, acting along the string towards A, and its weight $\boldsymbol{mg}$, acting vertically downwards.

The tangential component of the resultant force acting on the particle is $mg\sin\theta$, acting towards O. From the equation of motion in the tangential direction we have

$$m\ddot{s} = -mg\sin\theta$$

i.e. $ml\ddot{\theta} = -mg\sin\theta$

Now, considering only small angles, $\sin\theta \approx \theta$, the equation becomes

$$l\ddot{\theta} = -g\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\theta$$

$$\Rightarrow \ddot{s} = -\frac{g}{l}s$$

Comparing this equation with $\ddot{x} = -\omega^2 x$, we see that, for small oscillations, the motion of a simple pendulum is approximately simple harmonic with $\omega^2 = \dfrac{g}{l}$ and period $2\pi\sqrt{\dfrac{l}{g}}$

Worked examples

Example 1

A particle moves with simple harmonic motion with period 8 seconds and amplitude 80 centimetres. Find the maximum speed and acceleration of the particle and its speed when the particle is 50 centimetres from the centre of the motion.

Solution

$$T = \frac{2\pi}{\omega} = 8 \quad \Rightarrow 8\omega = 2\pi \quad \Rightarrow \omega = \frac{\pi}{4}$$

Maximum speed $= \omega a = \dfrac{\pi}{4} \times 0.8 = \dfrac{\pi}{5}\ \text{ms}^{-1}$.

Maximum acceleration $= \omega^2 a = \dfrac{\pi^2}{16} \times 0.8 = 0.05\pi^2\ \text{ms}^{-2}$ towards the centre.

$v^2 = \omega^2\left(a^2 - x^2\right)$

So when $x = 0.5$, $v^2 = \dfrac{\pi^2}{16}\left(0.8^2 - 0.5^2\right) = 0.39\dfrac{\pi^2}{16}$

Hence the speed of the particle, when 50 centimetres from the centre, is 0.49 ms^{-1}.

Example 2

A particle moving with simple harmonic motion has a speed of $4\sqrt{2}\ \text{ms}^{-1}$ and an acceleration of magnitude 8 ms^{-2} when it is 0.5m from the centre of oscillation. Find the period and amplitude of the motion, and the speed of the particle one fifth of a second after passing through the centre of oscillation.

Solution

$\ddot{x} = -\omega^2 x \quad \Rightarrow -8 = -0.5\omega^2 \quad \Rightarrow \omega^2 = 16 \quad \Rightarrow \omega = 4$

Thus period of motion $= T = \dfrac{2\pi}{\omega} = \dfrac{\pi}{2}$ seconds.

$v^2 = \omega^2\left(a^2 - x^2\right)$

$v = 4\sqrt{2}$ when $x = 0.5 \Rightarrow \left(4\sqrt{2}\right)^2 = 16\left(a^2 - 0.5^2\right)$

$\Rightarrow \dfrac{32}{16} = a^2 - 0.25$

$\Rightarrow a^2 = 2.25$

Thus the amplitude is 1.5 metres

Timing is from the centre, so we take $x = a\sin\omega t$

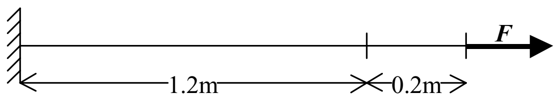
$x = a\sin\omega t \Rightarrow \dot{x} = a\omega\cos\omega t$

Thus, one fifth of a second after passing through the centre, the speed of the particle is $1.5 \times 4 \times \cos(4 \times 0.2) = 4.2 \text{ ms}^{-1}$

Example 3

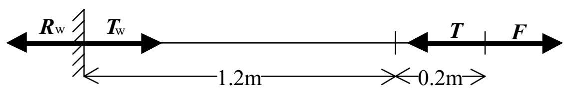
A light elastic string of natural length 1.2 metres is fixed at one end and stretched horizontally to a length of 1.4 metres by a force $\boldsymbol{F}$, as shown in the diagram.

If the magnitude of the force is 6 newtons, calculate the modulus of elasticity of the string.



Solution

If the tension in the string is T



The system is in equilibrium so $T = F = 6$

and $T = \dfrac{\lambda x}{l} \Rightarrow 6 = \dfrac{0.2\lambda}{1.2} \Rightarrow \lambda = \dfrac{6\times1.2}{0.2} = 36 \text{ N}$

(The forces $\boldsymbol{T}_w$ and $\boldsymbol{R}_w$ were included for completeness. Clearly $\boldsymbol{R}_w = \boldsymbol{T}_w = T = 6$)

Example 4

A simple pendulum which swings from one end of its path to the other end, in exactly one second, is called a seconds pendulum and it is said to beat seconds. A certain pendulum, at ground level where $g = 9.81$ ms^{-2}, beats exact seconds. When taken to another site where $g = 9.82$ ms^{-2}, by how many seconds in one whole day will it be wrong?

Solution

At ground level

$$T = 2\pi\sqrt{\frac{l}{g}}, g = 9.81 \text{ and } T = 2 \Rightarrow 2 = 2\pi\sqrt{\frac{l}{9.81}} \Rightarrow \pi\sqrt{l} = \sqrt{9.81}$$

$$\text{When } g = 9.82,\ T = 2\pi\sqrt{\frac{l}{9.81}} = 2\sqrt{\frac{9.81}{9.82}}$$

so time for one beat is $\sqrt{\frac{9.81}{9.82}}$ seconds

so number of beats in one day is now $\dfrac{24\times60\times60}{\sqrt{\frac{9.81}{9.82}}} = 86\ 444$, to nearest beat

so number of seconds gained in one day is $86\ 444 - 24 \times 60 \times 60 = 44$

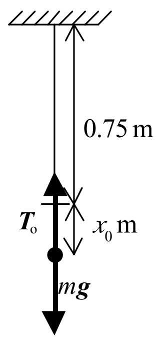
Example 5

A particle of mass m kg is suspended from one end of a string of modulus $3mg$ newtons and natural length 0.75 m, the other end of which is fixed.

(a) When hanging in equilibrium the string is stretched by x_o metres. Calculate x_o.

(b) The mass is now pulled down a further 0.2 metres and then released. Find the period of the subsequent motion and the maximum speed reached.

Solution

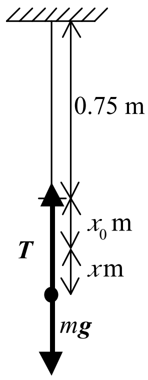


(a) $T_o = \dfrac{\lambda}{l}x_o$ and $T_o = mg$ (Equilibrium)

$$\Rightarrow \frac{3mgx_o}{0.75} = mg$$

$$\Rightarrow x_o = \frac{0.75}{3} = 0.25$$

(b) Consider the particle at x metres below the equilibrium position. The equation of motion (taking the downward sense as positive) gives



$$m\ddot{x} = mg - T$$

$$\Rightarrow \quad m\ddot{x} = mg - \frac{3mg}{0.75}(0.25 + x)$$

$$\Rightarrow \quad \ddot{x} = g - g - 4gx = -4gx$$

Comparing $\ddot{x} = -4gx$ with $\ddot{x} = -\omega^2 x$, motion is SHM with $\omega^2 = 4g$

Period $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2\sqrt{g}} = \dfrac{\pi}{\sqrt{g}}$ or 1 second, approximately.

$a = 0.2 \Rightarrow \omega a = 2\sqrt{g} \times 0.2 = 0.4\sqrt{g}$

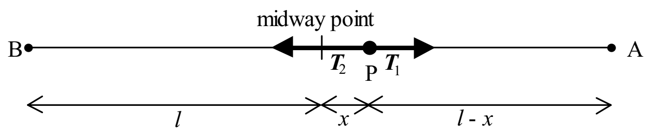
Thus maximum speed is 1.25 ms^{-1}

Example 6

An elastic string, of natural length l metres and modulus of elasticity λ newtons, is stretched between two fixed points A and B, distant $2l$ metres apart, on a smooth horizontal surface. A particle of mass m kilograms is attached to the midpoint M of the string. The particle is then pulled towards A through a distance $0.25l$ metres and then released. Show that the motion is simple harmonic and find the period of this motion. Find also the magnitude of the maximum acceleration of the particle.

Solution

The forces acting on the particle when it is x metres from the point midway between B and A are the tension T_1 in the 'AP' portion of the string and the tension T_2 in the 'BP' portion of the string.



'AP' portion of the string

Natural length $= \dfrac{l}{2}$ metres; stretched length $= l - x$ metres

Thus extension is $\dfrac{l}{2} - x$ metres, and

$$T_1 = \frac{\lambda}{\frac{l}{2}}\left(\frac{l}{2} - x\right) = \frac{2\lambda}{l}\left(\frac{l}{2} - x\right) = \frac{\lambda}{l}(l - 2x)$$

'BP' portion of the string

Natural length $= \dfrac{l}{2}$ metres; stretched length $= l + x$ metres

Thus extension is $\dfrac{l}{2} + x$ metres, and

$$T_2 = \frac{\lambda}{\frac{l}{2}}\left(\frac{l}{2} + x\right) = \frac{2\lambda}{l}\left(\frac{l}{2} + x\right) = \frac{\lambda}{l}(l + 2x)$$

The greatest distance the particle can be from the midway point is $\dfrac{l}{4}$ m and since $\dfrac{l}{4} < \dfrac{l}{2}$, neither portion of the string will go slack during motion.

Using the equation of motion, with $\overrightarrow{AB}$ the positive sense,

$$m\ddot{x} = T_1 - T_2 = \frac{\lambda}{l}(l - 2x) - \frac{\lambda}{l}(l + 2x) = -\frac{4\lambda}{l}x$$

Thus $\ddot{x} = -\dfrac{4\lambda}{ml}x$ and hence the motion is SHM with $\omega^2 = \dfrac{4\lambda}{ml}$

Period $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{ml}{4\lambda}} = \pi\sqrt{\dfrac{ml}{\lambda}}$

Magnitude of maximum acceleration $= \omega^2 a = \dfrac{4\lambda}{ml} \times \dfrac{l}{4} = \dfrac{\lambda}{m}$ ms^{-2}

Resources/examples

S&T	Chapter 17 (Simple Harmonic Motion)		
		Pages 434–41	Ex 17A, Pages 439–41
		Pages 441–9	Ex 17B, Pages 446–8
		Pages 449–54	Ex 17C (Harder examples)
		Pages 454–5	Ex 17D (Simple Pendulum)
		Pages 456–8	Ex 17E (Selected questions)
	Chapter 15 (Elasticity)		
		Pages 359–64	Ex 15A (Hooke's law)
RCS	Chapter 14 (Techniques of Dynamics)		
		Pages 274–8	(Avoid proofs on page 275 at this stage!) Ex 14C, Nos. 1–5, 9
		Pages 278–82	Ex 14D, Nos. 1–6, 10–16
	Chapter 10 (Elasticity)		
		Pages 185–7	Ex 10A, Nos. 1–7
		Pages 188–91	Ex 10B, Nos. 1–10
		Pages 191–3	Ex 10C, Nos. 1–6
TG	Chapter 14 (Simple Harmonic Motion)		
		Pages 264–71	Ex 14.1A, Pages 269–70
			Ex 14.1B, Pages 271–2
		Pages 274–6	Ex 14.2A, Nos. 2–10
Ex 14.2B,	Pages 276–7		
		Pages 277–82	Ex 14.3A, Nos. 1–5
			Ex 14.3B, Nos. 1–7
	Chapter 11 (Hooke's law)		
		Pages 201–4	Ex 11.1A, Nos. 1–7
			Ex 11.1B, Nos. 1–5
B&C	Chapter 12 (Simple Harmonic Motion)		
		Pages 382–92	(Avoid proof by differential equations, page 383, at this stage) Ex 12a, Page 388 Ex 12b Page 392
		Pages 393–5	Ex 12c, Page 396 (Simple Pendulum)
		Pages 396–412	Ex 10d, Pages 400–1 (Selected examples – avoid any involving energy at this stage)

Chapter 17 (Hooke's law)

Pages 199–208 (Omit worked examples 5 and 6 on pages 204 and 205)   Ex 7a, Pages 207–8

OG (Note – no Simple Harmonic Motion)

Chapter 8; (8.1:5 Hooke's law)

Pages 434–5   Ex 8.1:5, Nos. 1–7

SECTION 3

Content

- Know that force is the rate of change of momentum
- Know that impulse is the change in momentum
  i.e. $\boldsymbol{I} = m\boldsymbol{v} - m\boldsymbol{u} = \int \boldsymbol{F}dt$
- Understand the concept of conservation of linear momentum
- Solve problems on linear motion such as the motion of lifts, recoil of a gun, pile-drivers, etc.

Comments

The equation $\boldsymbol{F} = m\boldsymbol{a}$ is again involved here and equations of motion with constant acceleration could recur.

Teaching notes

Linear momentum

The linear momentum of a body of mass m, moving with velocity $\boldsymbol{v}$, is defined to be $m\boldsymbol{v}$. Since $\boldsymbol{v}$ is a vector quantity, then momentum is a vector quantity. When m is measured in kg and $\boldsymbol{v}$ in ms^{-1}, then the units of momentum are newton-seconds, i.e. Ns.

Force = rate of change of momentum

Newtons second law of motion, which gives the equation of motion $\boldsymbol{F} = m\boldsymbol{a}$, i.e. 'Force = mass × acceleration',

can be stated as:

'Force = mass × rate of change of velocity'

i.e. $\boldsymbol{F} = m\dfrac{d\boldsymbol{v}}{dt}$

i.e. $\boldsymbol{F} = \dfrac{d}{dt}(m\boldsymbol{v})$ for constant mass m.

Thus 'Force = rate of change of momentum'

Impulse

The effect that a force has on a body depends on the force and the length of time the force is applied.

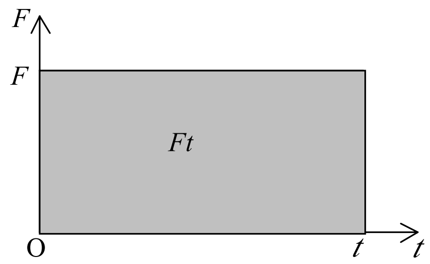
The impulse of a constant force

If a constant force is applied then the effect will be proportional to the time it is applied. For a constant force $\boldsymbol{F}$, $\boldsymbol{F}t$ is called the impulse of $\boldsymbol{F}$ during the interval t.

$$\begin{aligned} \boldsymbol{F}t &= m\boldsymbol{a} \times t && [\text{since } \boldsymbol{F} = m\boldsymbol{a}] \\ &= m(\boldsymbol{a}t) \\ &= m(\boldsymbol{v} - \boldsymbol{u}) && [\text{since } \boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t] \\ &= m\boldsymbol{v} - m\boldsymbol{u} \\ &= \textbf{change in momentum produced} \end{aligned}$$

The impulse of any force

From the graph of the magnitude, F, of the constant force $\boldsymbol{F}$ against t we see that the magnitude of the impulse, Ft, is the area beneath the graph of F.



That is **Impulse** $= \int_0^t \boldsymbol{F}dt$

For any force $\boldsymbol{F}$, we define the impulse in the interval t to be $\int_0^t \boldsymbol{F}dt$

As we noted above, $\boldsymbol{F} = m\dfrac{d\boldsymbol{v}}{dt}$ and hence

$$\int_0^t \boldsymbol{F}dt = \int_0^t m\frac{d\boldsymbol{v}}{dt}dt = m\int_{\boldsymbol{u}}^{\boldsymbol{v}} d\boldsymbol{v} = m[\boldsymbol{v}]_{\boldsymbol{u}}^{\boldsymbol{v}} = m\boldsymbol{v} - m\boldsymbol{u} = \textbf{change in momentum}$$

Thus the impulse, *I*, of any force *F* over an interval *t* is the change in momentum over the interval *t*,

i.e. $\quad \boldsymbol{I} = m\boldsymbol{v} - m\boldsymbol{u} = \int \boldsymbol{F}dt$

If the interval of time is from t_1 to t_2 then impulse $= \int_{t_1}^{t_2} \boldsymbol{F}dt$

The unit of impulse is the same as that for momentum and impulse is a vector quantity.

Conservation of linear momentum

When two bodies come into contact with one another, each exerts a force on the other and Newton's third law of motion states that the force exerted by body A on body B is equal and opposite to that exerted by B on A.

Consequently, the impulse exerted by A on B is equal and opposite to that exerted by B on A, and so the change in momentum of A is equal and opposite to the change in momentum of B.

This implies that the combined momentum of A and B remains unchanged, giving the result:

> *When no external forces act on a system, the total momentum of the system remains constant, i.e. when two bodies collide, the total linear momentum after the collision must equal the total linear momentum before the collision.*

This is the **principle of the conservation of linear momentum**.

Worked examples

Example 1

A ball of mass 0.3 kg is dropped from a height of 10 m onto smooth ground. It rebounds to a height of 5 m. Find the impulse exerted by the ground on the ball.

Solution

Impulse = change in momentum

'Falling ball': Using: $v^2 = u^2 + 2as$: $v^2 = 2 \times 9.8 \times 10 \Rightarrow v = 14$
Thus the speed of the ball immediately before impact is 14 ms^{-1}

'Rising ball': Using: $v^2 = u^2 + 2as$: $0 = u^2 + 2 \times 9.8 \times 5 \Rightarrow u = 9.9$
Thus speed of ball immediately after impact is 9.9 ms^{-1}

Taking downwards as the positive direction

Momentum before impact = $0.3 \times 14 = 4.2$ Ns

Momentum after impact = $-0.3 \times 9.9 = -2.97$ Ns

Thus Impulse = change in momentum = $4.2 + 2.97 = 7.7$ Ns

Example 2

A carriage of mass 12 000 kg, travelling at 0.4 ms^{-1}, is brought to a stop by buffers over a period of 3 seconds. Find the constant force exerted by the buffers on the carriage.

Solution

Force is constant so Ft = Impulse = $mv - mu$

Thus $3F = 12\,000(0 - 0.4)$

Now, $3F = 12\,000(0 - 0.4) \Rightarrow 3F = -4800 \Rightarrow F = -1600$

Thus the buffers exert a force of 1600 newtons to stop the carriage.

Example 3

An aircraft of mass 2000 kg taxis down a runway under the action of a resultant force whose magnitude is given by $6(1000 - t^2)$ newtons, where t is the time in seconds from the start. The plane takes off after 40 seconds. Find the impulse on the plane during these 40 seconds and the speed of the plane at the instant of take-off.

Solution

There is a variable force so we use $\boldsymbol{I} = \int_0^t \boldsymbol{F}dt$

Impulse =

$$\int_0^{40} (6000 - 6t^2)dt = [6000t - 2t^3]_0^{40} = 240\,000 - 2 \times 64\,000 = 112\,000$$

Thus impulse on the plane over the interval of 40 seconds is 112 000 Ns

But Impulse = change in momentum = m(v – u)

so 112 000 = 2000(v – 0)
$\Rightarrow v = 56$
Thus speed on take-off is 56 ms^{-1}.

Example 4
A bullet of mass 50 g is fired from a gun of mass 2 kg. Given that the bullet leaves the gun with a horizontal velocity of 360 ms^{-1}, find the initial speed of recoil of the gun.

Solution
Using the Principle of Conservation of Momentum
Momentum before firing = 0
Momentum immediately after firing = $0.05 \times 360 + 2 \times (-v)$
Hence, $18 - 2v = 0$
$\Rightarrow v = 9$
Thus gun recoils with a speed of 9 ms^{-1}

Example 5
Referred to rectangular axes Ox, with unit vector $\boldsymbol{i}$, and Oy with unit vector $\boldsymbol{j}$, a body of mass 3 kg, moving with velocity $(4\boldsymbol{i} - \boldsymbol{j})\text{ms}^{-1}$, collides and coalesces with a body of mass 2 kg moving with velocity $(-\boldsymbol{i} + 4\boldsymbol{j})\text{ms}^{-1}$. Find the magnitude and direction of the combined body immediately after the collision.

Solution
By the Principle of Conservation of Momentum
Momentum before impact = Momentum after impact

$\Rightarrow$ $3(4\boldsymbol{i} - \boldsymbol{j}) + 2(-\boldsymbol{i} + 4\boldsymbol{j}) = 5(u\boldsymbol{i} + v\boldsymbol{j})$
[where velocity of coalesced particles is $(u\boldsymbol{i} + v\boldsymbol{j})$]
$\Rightarrow$ $12\boldsymbol{i} - 2\boldsymbol{i} = 5u\boldsymbol{i}$ and $-3\boldsymbol{j} + 8\boldsymbol{j} = 5v\boldsymbol{j}$
$\Rightarrow$ $5u = 10$ and $5v = 5$
$\Rightarrow$ $u = 2$ and $v = 1$

Thus velocity immediately after impact is $2\boldsymbol{i} + \boldsymbol{j}$

Magnitude $= \sqrt{4 + 1} = \sqrt{5}\,\text{ms}^{-1}$. Direction is $\tan^{-1}\dfrac{1}{2} = 26.6°$ to the x-axis.

Example 6

Particle P, of mass 2 kg, moving with speed 8 ms^{-1}, collides with particle Q, of mass 3 kg at rest on a smooth horizontal surface. After the impact P continues to move in the same direction but with speed 2 ms^{-1}.
Find the speed with which Q starts to move.

Solution

By the Principle of Conservation of Momentum

Momentum before impact = Momentum after impact

$\Rightarrow \quad 2 \times 8 + 3 \times 0 = 2 \times 2 + 3v$ [where v is the speed of Q after impact]

$\Rightarrow \quad 16 = 4 + 3v$

$\Rightarrow \quad 3v = 12$

$\Rightarrow \quad v = 4$

Thus speed of Q immediately after impact is 4 ms^{-1}.

Resources/examples

S&T Chapter 14 (Momentum and Impulse)

Pages 341–4	Ex 14A
Pages 345–6	Ex 14B
Pages 347–8, 350, 351–4	Ex 14C (omit questions about energy at this stage)
Page 355	Ex14D, Nos. 1–6 (omit questions about energy at this stage)

RCS Chapter 7 (Momentum and Impulse)

Pages 114–16	Ex 7A
Pages 117–18	Ex 7B
Page 119	Ex 7C
Pages 120–1	Ex 7D
Pages 122–4	Ex 7E

TG Chapter 9 (Momentum and Collisions)

Pages 131–2	Ex 9.1A and 9.1B
Pages 134–5	Ex 9.2A and 9.2B
Pages 139–41, Page 143	Ex 9.3A and Ex 9.3B (selected examples)
Pages 146–9	Ex 9.4A and 9.4B
Pages 152–5	Ex 9.5A and 9.5B

B&C Chapter 8 (Momentum and Direct Impact)

Pages 232–5	Ex8a, page 236 (Impulse)
Pages 237–40	Ex8b, page 241 (Conservation of Momentum)

OG Chapter 8, 8.5 (Momentum)

Pages 488–9	Ex 8.5:1, Page 490 (selected examples)
Pages 491–3	Ex 8.5:2 (selected examples, omit energy at this stage and connected particles)

SECTION 4

Content

- Know the meaning of the terms work and power
- Understand the concept of work
- Calculate the work done by a constant force in one and two dimensions, i.e. $W = Fd$ (one dimension); $W = \boldsymbol{F}.\boldsymbol{d}$ (two dimensions)
- Calculate the work done in rectilinear motion by a variable force using integration, i.e. $\int \boldsymbol{F}.\boldsymbol{i}dx$; $W = \int \boldsymbol{F}.\boldsymbol{v}dt$, where $\boldsymbol{v} = \frac{dx}{dt}\boldsymbol{i}$
- Understand the concept of power as the rate of doing work, i.e. $P = \frac{dW}{dt} = \boldsymbol{F}.\boldsymbol{v}$ (constant force), and apply this in practical examples

Teaching notes

Work done

Whenever the point of application of a force moves through a certain distance, work is done, provided the direction of motion is not perpendicular to the force.

Consider the point of application of a **constant** force $\boldsymbol{F}$ newtons moving from the point A to the point B, where $\overrightarrow{AB} = \boldsymbol{d}$, distance being measured in metres. Then the work done, W, is defined to be

$$W = \boldsymbol{F}.\boldsymbol{d}$$

If the angle between $\boldsymbol{F}$ and $\boldsymbol{d}$ is θ, then $W = Fd\cos\theta$.

Note that work is a scalar quantity, and that the work done is not dependent on the path of the force, but only on its end point. Also work can be done by or against a force.

$$\begin{aligned} W &= Fd\cos\theta \\ &= F\cos\theta d \\ &= \text{component of force in the direction of the displacement} \times \text{distance moved.} \end{aligned}$$

In particular, if the distance moved is in the direction of the force then
$W = Fd$
and if the displacement is perpendicular to the force then $W = 0$.

The unit of work is the joule or Newton metre (1 J = 1 Nm).

Now consider a **variable** force $\boldsymbol{F}$ newtons moving in a straight line.

In the case where $\boldsymbol{F}$ is a function of its position x on the line of action then $\boldsymbol{F} = \boldsymbol{F}(x)$. If the point of action of the force moves from $x = x_0$ to $x = x_1$, then consider the line divided into n equal parts of length

$$\partial x = \frac{x_1 - x_0}{n}.$$

If n is large then ∂x is small and the force will be approximately constant over each interval. If the force is $\boldsymbol{F}_r$ in the rth interval then the work done in this interval is $\boldsymbol{F}_r.\partial x\boldsymbol{i}$ where $\boldsymbol{i}$ is the unit vector in the direction from $x = x_0$ to $x = x_1$.

Thus the total work done as the point of action moves from $x = x_0$ to $x = x_1$ is

$$W = Lim_{\partial x \to 0}\left(\sum_{x_0}^{x_1} \boldsymbol{F}_r.\partial x\boldsymbol{i}\right) = \int_{x_0}^{x_1} \boldsymbol{F}.\boldsymbol{i}dx$$

or $W = \int_{x_0}^{x_1} F(x)dx$

(The scalar form being used since the function is one-dimensional.)

In the case where $\boldsymbol{F}$ is a function of time, then $\boldsymbol{F} = \boldsymbol{F}(t)$.
If the time interval over which $\boldsymbol{F}$ acts is from $t = t_0$ to $t = t_1$

$$W = \int_{x_0}^{x_1} F(t)dx = \int_{t_0}^{t_1} F(t)\frac{dx}{dt}dt = \int_{t_0}^{t_1} Fvdt$$

The vector form here is

$$W = \int_{t_0}^{t_1} \boldsymbol{F}.\boldsymbol{v}dt \text{ where } \boldsymbol{v} = \frac{dx}{dt}\boldsymbol{i}$$

This will recur when power is considered.

Resources/examples

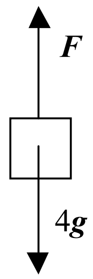
Please note that the notes on Resources/examples for Sections 4 and 5 are integrated and appear after Section 5 (on pages 61–2).

Worked examples

Example 1

Find the work done against gravity when a body of mass 4 kg is raised through a vertical distance of 8 m.

Solution



The vertical force required is given by $F = 4g$ (scalar form is being used as motion is in a straight line)

So $W = F\mathrm{d}$

$= 4g \times 8$

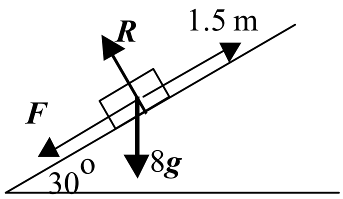
$= 32g$

$= 313.6$ J

Example 2

A body of mass 8 kg is pulled at constant speed through a distance of 1.5 m up the line of greatest slope of a rough plane, which is inclined at an angle of 30° to the horizontal. Find the total work done against gravity and friction, given that the coefficient of friction between the body and the surface is 0.4.

Solution



Work done against gravity

$=$ force $\times$ vertical distance moved

$= 8g \times 1.5 \times \sin 30°$

$= 58.8$ J

Resolving perpendicular to the plane

$R = 8g\cos 30° = 67.9$ N

So $F = \mu R = 0.4 \times 67.9 = 27.2$ N

Work done against friction $= 27.2 \times 1.5 = 40.8$ J

Total work done $= 58.8 + 40.8 = 99.6$ J

Note that this is the work done **by** the pulling force.

Example 3

Find the work done by a constant force $\boldsymbol{F} = 2\boldsymbol{i} + \boldsymbol{j}$ moving from point A with position vector $\boldsymbol{i} + 2\boldsymbol{j}$ to a point B with position vector $5\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{i}$ and $\boldsymbol{j}$ being unit vectors in the directions of the x and y axes respectively.

Solution

$$\overrightarrow{AB} = (5\boldsymbol{i} - \boldsymbol{j}) - (\boldsymbol{i} + 2\boldsymbol{j}) = 4\boldsymbol{i} - 3\boldsymbol{j}$$
$$W = \boldsymbol{F}.\boldsymbol{d} = (2\boldsymbol{i} + \boldsymbol{j}).(4\boldsymbol{i} - 3\boldsymbol{j}) = 8 - 3 = 5\text{ J}$$

Example 4

A particle is constrained to move along the x-axis under the action of a force $(2x + 3)\boldsymbol{i} + 5\boldsymbol{j}$ newtons, where $\boldsymbol{i}$ is the unit vector in the direction of the x-axis, and $\boldsymbol{j}$ is the unit vector in the perpendicular direction. Calculate the work done when the point of application of the force moves from the origin to $x = 4$, where distances are measured in metres.

Solution

The component $5\boldsymbol{j}$ does no work since it is perpendicular to the direction of motion.

$$W = \int_0^4 (2x + 3)\,dx = [x^2 + 3x]_0^4 = 16 + 12 = 28\text{ J}$$

Example 5

A particle P, of mass 3 kg, moves in a horizontal straight line under the influence of a variable force $\boldsymbol{F}$. At time t seconds, the velocity $\boldsymbol{v}$ ms^{-1} of P is given by $\boldsymbol{v} = (2t + t^2)\boldsymbol{i}$, where $\boldsymbol{i}$ is the unit vector in the direction of motion. Find

(a) the acceleration, $\boldsymbol{a}$ ms^{-2}, of P at time t seconds
(b) the work done by $\boldsymbol{F}$ in the first two seconds.

Solution

(a) $\boldsymbol{a} = \dfrac{dv}{dt} = \dfrac{d}{dt}(2t + t^2)\boldsymbol{i} = (2 + 2t)\boldsymbol{i}$

(b) $\boldsymbol{F} = 3\boldsymbol{a} = (6 + 6t)\boldsymbol{i}$

$$W = \int_0^2 \boldsymbol{F}.\boldsymbol{v}dt = \int (6 + 6t)(2t + t^2)dt = (12t + 18t^2 + 6t^3)dt$$
$$= \left[6t^2 + 6t^3 + \frac{3}{2}t^4\right]_0^2 = 24 + 48 + 24 = 96\text{ J}$$

Power

Power is the rate at which a force does work.
The unit of power is the **watt**. When 1 Joule of work is done in 1 second, the power is 1 watt.
1 **kilowatt** = 1000 **watts**
Power, like work, is a scalar quantity.

When the work done is W, the power P is then given by $P = \dfrac{dW}{dt}$

The work done by a constant force $\boldsymbol{F}$ is then $W = \boldsymbol{F}.\boldsymbol{d}$ where $\boldsymbol{d}$ is the distance moved.

So $P = \dfrac{d}{dt}(\boldsymbol{F}.\boldsymbol{d})$

$= \boldsymbol{F}.\dfrac{d\boldsymbol{d}}{dt}$ since $\boldsymbol{F}$ is constant

$= \boldsymbol{F}.\boldsymbol{v}$ where $\boldsymbol{v}$ is the velocity

When the force and the velocity are in the same direction this reduces to $P = Fv$.

When a car is being driven along a road the force propelling it, $\boldsymbol{F}$ newtons, is supplied by the engine. When this engine is working at a constant rate of P watts, then

$P = Fv$ so $F = \dfrac{P}{v}$

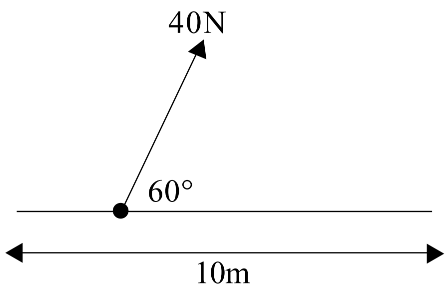
This force is called the traction force exerted by the engine when the car is travelling at v ms^{-1}.

Worked examples

Example 1

A particle is moving along a straight wire under the action of a constant force of 40 N. The force acts at an angle of 60° to the wire and moves the particle a distance of 10 m in 4 seconds. Find the average rate at which the force is working.

Solution



Component of force in direction of wire = 40cos60°= 20 N
Work done by the force = 20 × 10 = 200 J

Average rate of working = $\dfrac{200}{4}$ = 50 W

Example 2

A constant force, $\boldsymbol{F} = 5\boldsymbol{i} - 2\boldsymbol{j}$ N, acts on a particle, moving it in a straight line from a point A, with position vector $-\boldsymbol{i} + 2\boldsymbol{j}$, to a point B, with position vector $2\boldsymbol{i} + 4\boldsymbol{j}$, in 5 seconds, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the directions of the x and y axes. Find the average rate at which $\boldsymbol{F}$ is working.

Solution

$W = \boldsymbol{F}.\boldsymbol{d}$ [where $\boldsymbol{d} = 2\boldsymbol{i} + 4\boldsymbol{j} - (-\boldsymbol{i} + 2\boldsymbol{j}) = 3\boldsymbol{i} + 2\boldsymbol{j}$]
$= (5\boldsymbol{i} - 2\boldsymbol{j}).(3\boldsymbol{i} + 2\boldsymbol{j}) = 15 - 4 = 11$ J

Average rate of working = $\dfrac{11}{5}$ = 2.2 W

Example 3

A car of mass 1.5 tonnes travels at a uniform speed of 30 ms^{-1} along a straight horizontal road against resistance of 40 N per tonne. Find the power generated by the engine. If the driver now decides to accelerate, what is the maximum acceleration he could achieve given that the car has a 60 kW engine?

Solution

Resistive force = $40 \times 1.5 = 60$ N

Since speed is uniform, tractive force = resistance

$P = Fv = 60 \times 30 = 1800$ W

Surplus power available for acceleration = $6000 - 1800 = 4200$ W

So maximum accelerating force at 30 ms^{-1} = $\dfrac{4200}{30} = 140$ N

If the maximum acceleration at 30 ms^{-1} is a, then using $F = ma$,

$140 = ma$

$140 = 1500a$

$a = \dfrac{140}{1500} = 0.09$ ms^{-2}

Thus maximum acceleration available at 30 ms^{-1} is 0.09 ms^{-2}

Example 4

A car of mass 2000 kg travels at a constant speed of 30 ms^{-1} along a straight level road, against a constant resistance of 600 N. Find the power generated by the engine. If the engine continues to work at the same rate, against the same resistance, find the maximum speed at which it can ascend an incline of $\sin^{-1} \dfrac{1}{50}$.

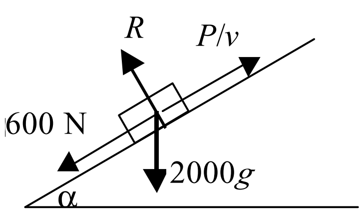
Solution

Since speed is uniform there is no acceleration, so tractive force = resistance

Hence $\dfrac{P}{v} = 600$

so $\dfrac{P}{30} = 600$

thus $P = 18\,000$ W = 18 kW



Up incline maximum speed occurs when there is no acceleration.

If maximum speed is v then

$$\frac{P}{v} = 600 + 2000g \sin\left(\sin^{-1}\frac{1}{50}\right)$$

$$\Rightarrow \frac{18000}{v} = 600 + 2000 \times 9.8 \times \frac{1}{50}$$

$$\Rightarrow \frac{18000}{v} = 992$$

$$\Rightarrow v = \frac{18000}{992} = 18.1 \text{ ms}^{-1}$$

SECTION 5

Content

- Know the meaning of the terms potential energy and kinetic energy
- Understand the concept of energy and the difference between kinetic (E_k) and potential (E_p) energy
- Know that $E_k = \frac{1}{2}mv^2$
- Know that the potential energy associated with
  a. a uniform gravitational field is $E_p = mgh$
  b. Hooke's law is $E_p = \frac{1}{2}k(\text{extension})^2$
  c. Newton's inverse square law is $E_p = \frac{GmM}{r}$
- Understand and apply the work–energy principle
- Understand the meaning of conservative forces like gravity, and non-conservative forces like friction
- Know and apply the energy equation $E_k + E_p =$ constant, including to the situation of motion in a vertical circle

Teaching notes

Energy

The energy of a body is the capacity of the body to do work. It is a scalar quantity and the unit of energy is the same as that of work, namely the joule. There are many different forms of energy but only two are considered here, namely kinetic and potential energy. When a force does work on a body it changes the energy of the body.

Kinetic energy

The ***kinetic energy*** of a body is the energy it possesses by virtue of its **motion**.
Consider a constant horizontal force $\boldsymbol{F}$ newtons acting on a body of mass m kg which was initially at rest on a smooth horizontal surface. After the force has moved the body through a distance s metres in a straight line across the surface, the body will have acquired a speed of v ms^{-1}. The work done by the force in doing this is a measure of the increase in the kinetic energy of the body.

Work done = Fs
But $F = ma$, where a is the acceleration of the body, and $v^2 = u^2 + 2as$

so $a = \dfrac{v^2 - 0}{2s}$ and $F = \dfrac{mv^2}{2s}$

Hence, work done $= \dfrac{mv^2}{2s} \times s = \dfrac{mv^2}{2}$ joules

$\dfrac{mv^2}{2}$ is the kinetic energy of the body of mass m when its speed is v.

Kinetic energy is often denoted by the letter T, or by E_k.

From $T = \dfrac{1}{2}mv^2$ and

differentiating with respect to time we obtain

$$\dot{T} = mv\frac{dv}{dt} = mav = Fv = P$$

Thus the rate of change of kinetic energy of a body is equal to the power of the force acting on the body.

Potential energy

Potential energy is the energy a body possesses by virtue of its **position**. In this unit we consider gravitational potential energy and the potential energy of a spring or elastic string.

Gravitational potential energy

Consider first a body of mass m kg being raised a distance of h metres vertically under gravity from the surface of the Earth. (If h is small compared with the radius of the Earth then the gravitational force may be assumed to be constant, i.e. we have a **uniform gravitational field**.) The work done against gravity is then mgh joules where g ms^{-2} is the magnitude of the acceleration due to gravity at the surface of the Earth. This is a measure of the increase in the potential energy of the body. Potential energy is sometimes denoted by V and also by E_p. So $E_p = mgh$.

Note

There is no actual zero of potential energy and any arbitrary level may be used as a reference point to measure the change in potential energy of a body.

In general if a particle is moving freely under constant gravity its equation of motion is $m\ddot{z} = -mg$

and so $\int m\ddot{z}dz = \int -mgdz = -mgz + C$

$$\text{Now } \ddot{z} = \frac{d\dot{z}}{dt} = \frac{d\dot{z}}{dz}\frac{dz}{dt} = \frac{d\dot{z}}{dz}\dot{z} = \dot{z}\frac{d\dot{z}}{dz}$$

$$\therefore \quad \int m\ddot{z}dz = -mgz + C$$

$$\Rightarrow \quad \int m\dot{z}\frac{d\dot{z}}{dz}dz = -mgz + C$$

$$\Rightarrow \quad \int m\dot{z}d\dot{z} = -mgz + C$$

$$\Rightarrow \quad \frac{1}{2}m\dot{z}^2 = -mgz + C$$

$$\Rightarrow \quad \frac{1}{2}m\dot{z}^2 + mgz = C$$

$$\Rightarrow \quad E_k + E_p = \text{constant}$$

Taking the more general case, and using Newton's Inverse Square Law, we have

Gravitational force $= \dfrac{GmM}{r^2}$

and the equation of motion of the body of mass m is $m\ddot{r} = \frac{GmM}{r^2}$

(where r is measured from particle of mass m).

Proceeding as above we have

$$\int m\ddot{r}dr = \int \frac{GmM}{r^2} = -\frac{GmM}{r} + C$$

$$\Rightarrow \quad \int m\dot{r}d\dot{r} = -\frac{GmM}{r} + C$$

$$\Rightarrow \quad \frac{1}{2}m\dot{r}^2 = -\frac{GmM}{r} + C$$

$$\Rightarrow \quad \frac{1}{2}m\dot{r}^2 + \frac{GmM}{r} = C$$

$\frac{1}{2}m\dot{r}^2$ is the kinetic energy of the particle and $\frac{GmM}{r}$ is the potential energy of the particle.

Again we see that $E_k + E_p$ = constant

Since $E_k + E_p$ = constant, **mechanical energy is conserved** and gravitation is an example of a **conservative force**.

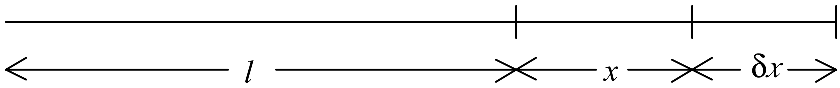
Conservative and non-conservative forces

As an object moves from A to B under constant gravity, the work done by its weight is equal to the difference between the potential energies at B and A, i.e. it depends only on the positions of A and B and not on the path taken from A to B. This is typical of conservative forces. Other examples of conservative forces include the tension in a spring. When frictional forces act on a body some of the energy is converted into heat and the mechanical energy is reduced. Friction is an example of a **non-conservative** force. The work done by friction on a body as it moves from a position A to a position B depends on the length of the path taken, the longer the path the greater the work done.

Elastic potential energy

The other case of potential energy considered is the energy stored in an elastic string (or spring) when it has been stretched a distance x.

The tension in the string is then $\frac{\lambda x}{l}$, l being the natural length of the string and λ is the modulus of elasticity.



The work done in stretching it a further small distance δx is thus $\dfrac{\lambda x}{l}\delta x$

Thus, the work done in stretching the string from its natural length, l, to the length $l + x$ is given by

$$\lim_{\delta x \to 0} \sum \frac{\lambda x}{l}\delta x = \int_0^x \frac{\lambda}{l} x\,dx = \frac{1}{2}\frac{\lambda}{l}x^2 = \frac{1}{2}\frac{\lambda}{l}\ (\text{extension})^2$$

In the case of a spring this would be $\frac{1}{2}\boldsymbol{k}$ (extension)^2, $\boldsymbol{k}$ being the stiffness constant.

If the spring were to return to its natural length, the tension would do a positive amount of work $\frac{1}{2}kx^2$. This expression is called the elastic potential energy of the spring.

Worked examples

(Note: Some of the questions tackled here by the work/energy principle can be answered by other means.)

Example 1

A body of mass 2 kg is pulled in a straight line across a smooth horizontal surface by a constant, horizontal force $\boldsymbol{P}$ newtons. The body passes through a point A with speed 4 ms^{-1}, then, 5 metres further along, passes a point B with speed 6 ms^{-1}. Find the magnitude of the force $\boldsymbol{P}$.

Solution

K.E. at A $= \frac{1}{2} \times 2 \times 4^2 = 16\,J$

K.E. at B $= \frac{1}{2} \times 2 \times 6^2 = 36\,J$

Work done = change in K.E. = 36 – 16 = 20 J

But work done $= P \times 5$ so $P = 4$ N

Example 2

A point A is 20 m vertically above a second point B. A body, of mass 4 kg, is released from rest at A and falls vertically against a constant resistance of 24 N. Find the speed of the body at B.

Solution

(Taking B as the zero of P.E.)

Total Energy at A = K.E + P.E. $= 0 + 4g \times 20 = 80g$ J

Total Energy at B = K.E + P.E. $= \frac{1}{2} \times 4v^2 = 2v^2$ J

The body does work against the resistance in travelling from A to B and so loses energy.

Work done against resistance = loss in energy $= 80g - 2v^2$

Thus, $24 \times 20 = 80g - 2v^2$

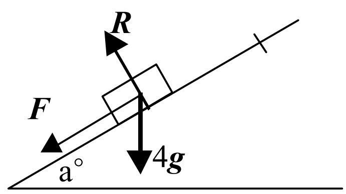
$\Rightarrow \quad 2v^2 = 784 - 480 = 304$

$\Rightarrow \quad v^2 = 152$

$\Rightarrow \quad v = 12.3\ ms^{-1}$

Example 3

From the foot of a rough, inclined plane, a body of mass 4 kg is projected, with speed 6 ms^{-1}, up the line of greatest slope. The plane is inclined at an angle of $\sin^{-1}\frac{3}{5}$ to the horizontal and the coefficient of friction between the body and the plane is $\frac{3}{8}$.



Find the distance the body travels up the plane before first coming to rest.

Solution

Take the foot of the plane as the zero of P.E.

Let d metres be the distance travelled up the plane.

$F = \mu R$ and, resolving perpendicular to the plane, $R = 4g\cos a° = 4g \times \frac{4}{5}$

So $F = \frac{3}{8} \times 4g \times \frac{4}{5} = \frac{6g}{5}$ N

Total energy at the start = K.E. + P.E. = $\frac{1}{2} \times 4 \times 6^2 + 0 = 72$ J

Total energy at highest point reached = K.E. + P.E. = $0 + 4g \times d\sin a°$

$$= 4g \times \frac{3}{5}d$$

$$= 23.52d \text{ J}$$

Work done against friction = loss in energy

$$\frac{6g}{5} \times d = 72 - 23.52d$$

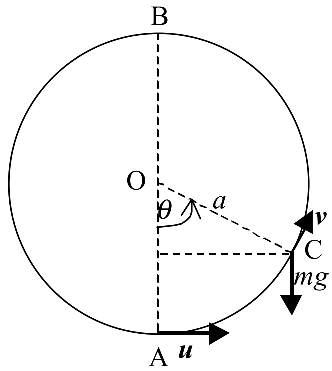
$\Rightarrow$ $11.76d + 23.52d = 72$

$\Rightarrow$ $35.28d = 72$

$\Rightarrow$ $d = 2.04$ m

The principal of Conservation of Energy is particularly useful in dealing with **motion in a vertical circle** and two examples are given here.

Example 4
A bead of mass m is threaded onto a smooth circular wire, of radius a, fixed in a vertical plane. The bead is projected from the lowest point, A, of the wire with speed u.



Find
(a) an expression for the speed v of the bead when it has reached the point C, where $\angle AOC = \theta$, as shown
(b) the least value of u that will allow the bead to execute complete revolutions.

Solution
(a) Taking A as the zero of P.E. the energy equation gives
Total energy at A = Total energy at C

$$\Rightarrow \frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mg(a - a\cos\theta)$$
$$\Rightarrow v^2 = u^2 - 2ga(1 - \cos\theta)$$

(b) For complete revolutions the speed at B must be positive
i.e. $v > 0$ when $\theta = 180°$

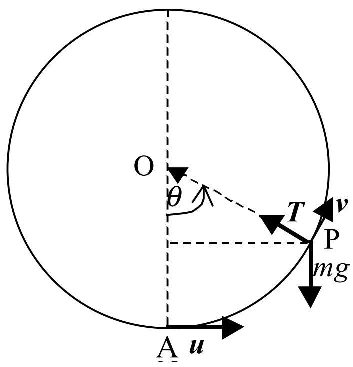
$$\Rightarrow u^2 - 2ga(1 - \cos 180°) > 0$$
$$\Rightarrow u^2 - 4ga > 0$$
$$\Rightarrow u^2 > 4ga$$
$$\Rightarrow u > 2\sqrt{ga}$$

Example 5
A particle P, of mass m, is suspended from a fixed point O by a light inextensible string of length a. When P is vertically below O it is given a horizontal velocity of magnitude u. Assuming that the string remains taut, find the speed of the particle when OP makes an angle θ with the downward vertical and find an expression for the tension, $\boldsymbol{T}$, in the string in terms of u, a, θ and g. Determine the condition for P to describe complete circles.

Solution

Taking A to be level of zero P.E.

Energy at A = Energy at P



$$\Rightarrow \frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mg(a - a\cos\theta)$$
$$\Rightarrow v^2 = u^2 - 2ga(1 - \cos\theta)$$
$$\Rightarrow v = \sqrt{u^2 - 2ga(1 - \cos\theta)}$$

Since P moves in a circle of radius a, there is an acceleration towards O of $\frac{v^2}{a}$ and so, resolving mg into its radial and transverse components, the equations of motion in the radial direction is

$$T - mg\cos\theta = \frac{mv^2}{a} \Rightarrow T = \frac{mv^2}{a} + mg\cos\theta$$

Substituting for v gives

$$T = \frac{m}{a}\left[u^2 - 2ga(1 - \cos\theta)\right] + mg\cos\theta$$
$$\Rightarrow T = \frac{mu^2}{a} - 2mg(1 - \cos\theta) + mg\cos\theta$$
$$\Rightarrow T = \frac{mu^2}{a} - 2mg + 3mg\cos\theta$$
$$\Rightarrow T = \frac{m}{a}\left[u^2 - ga(2 - 3\cos\theta)\right]$$

The particle will describe complete circles provided $T \geq 0$ for all θ, so that the string never goes slack.

Since T is at a minimum when $\theta = \pi$, then
$T \geq 0$ when $\theta = \pi$

$$\Rightarrow u^2 - ga(2 + 3) \geq 0$$
$$\Rightarrow u^2 - 5ga \geq 0$$
$$\Rightarrow u \geq \sqrt{5ga}$$

Notice that the minimum speed necessary for complete circles is greater here than in the previous example when the bead could not leave the circular path.

If $u < \sqrt{5ga}$, then at some stage the string will go slack and the particle will travel as a projectile, freely under gravity for a time. (Until the string goes taut again.)

Examples can also include a body on the inside, or the outside, of a circular drum.

Resources/examples

S&T Chapter 11 (Work, Energy and Power)

Pages 250–2	Ex 11A, Pages 252–3
Pages 262–4	Ex 11E, Page 265
Pages 266–7	Ex 11F, Pages 268–9
Pages 269–70	Ex11G, Pages 271–2
Pages 253–4	Ex 11B, Page 255
Pages 255–6	Ex 11C, Pages 257–8
Pages 258–9	Ex 11D, Pages 260–1
Pages 272–6	Ex 11H (selected examples)

Chapter 16 (Use of Calculus)

Pages 421–2	Ex 16F, Pages 423–4
Pages 427–32	Ex 16H (selected examples)

RCS Chapter 6 (Work, Energy and Power)

Pages 95–6	Ex 6A, Pages 97–8
	Ex 6B, Pages 106–7; Ex 6F
Pages 108–9	Ex 6G, Page 110
Pages 99–101	Ex 6C, Pages 102–3
	Ex 6D, Pages 104–5; Ex 6E
Pages 112–13	Examination Questions (selected examples)

Chapter 14 (Techniques of Dynamics)

Pages 268–9	Ex 14A, Pages 269–70
Pages 271–2	Ex 14B, Pages 273–4

TG Chapter 10 (Energy, Work and Power)

Pages 170–2	Ex 10.2A, Pages 172–3
	Ex 10.2B, Pages 174–5
Pages 175–7	Ex 10.3A, Pages 178–9
	Ex 10.3B, Pages 179–80
Pages 181–2	Ex 10.4A, Pages 183–4
	Ex 10.4B, Pages 185–6
Pages 187–8	Ex 10.5A, Page 189
	Ex 10.5B, Pages 190–1
Pages 191–3	Ex 10.6A, Page 194
	Ex 10.6B, Pages 195–6
Pages 196–8	Consolidation Exercises (selected examples)

Chapter 11 (Energy and Work: Variable Forces and Scalar Products)

Pages 205–8	Ex 11.2A, Pages 208–9
	Ex 11.2B, Pages 210–11 (selected examples)

	Pages 211–12	Ex 11.3A, Pages 213–14
		Ex 11.3B, Pages 215–16 (selected examples)
	Pages 216–17	Ex 11.4A, Pages 218–19
		Ex 11.4B, Pages 219–20
	Pages 221/2	Consolidation Examples (selected examples)
	Chapter 13 (Circular Motion with Variable Speed)	
	Pages 252–6	Ex 13.1A, Page 256
		Ex 13.1B, Pages 256–7 (selected examples)
	Page 260	Consolidation Exercises
B&C	Chapter 6 (Work and Power)	
	Pages 181–5	Ex 6a, Pages 185–6
	Pages 186–92	Ex 6b, Pages 192–3
	Pages 195–8	Ex 6
	Chapter 11 (General Motion of a Particle)	
	Pages 366–70	Ex 11j, Nos. 1–4
		Ex 11k, Page 370
	Chapter 7 (Energy)	
	Pages 208–12	Ex 7b, Page 212
	Pages 213–17	Ex 7c, Page 218
	Pages 218–23 (omit Ex 2)	Ex 7d, Pages 223–4;
		Ex 7, Pages 226–31 (selected examples)
OG	Chapter 8 (Forces on a particle)	
	8.4 Work and Energy; Power	
	Pages 470–2	Ex 8.4:1, Page 473
	Pages 474–5	Ex 8.4:2, Page 476
	Pages 476–9	Ex 8.4:3, Pages 480–3 (selected examples)
	Pages 483–4	Ex 8.4:4, Pages 485–7

SECTION 6

Content

- Know that $a = v\dfrac{dv}{dx}$ as well as $\dfrac{dv}{dt}$
- Use Newton's law of motion $\boldsymbol{F} = m\boldsymbol{a}$, to form first order differential equations to model practical problems, where the acceleration is dependent on the displacement or velocity,

  i.e. $\dfrac{dv}{dt} = f(v), v\dfrac{dv}{dx} = f(x), v\dfrac{dv}{dx} = f(v)$
- Solve such equations by the method of separation of variables
- Derive the equation $v^2 = \omega^2(a^2 - x^2)$ by solving $v\dfrac{dv}{dx} = -\omega^2 x$
- Know the meaning of the terms terminal velocity, escape velocity and resistance per unit mass, and solve problems involving differential equations and incorporating any of these terms or making use of $F = \dfrac{P}{v}$

Comments

It may be necessary to teach the solution technique of separation of variables, depending on the mathematical background of the students. Examples will be straightforward with integrals which are covered in Mathematics 1 and 2 at Advanced Higher. If more complex, then the anti-derivative will be given.

This section can involve knowledge and skills from other topics within this unit.

Teaching notes

Essentially, if the acceleration is not constant, then Newton's equations of motion cannot be used and the use of calculus is the appropriate technique. This cannot be stressed too strongly to candidates and has already been done so in Mechanics 1, when examples involving acceleration as a function of time were covered. There, integrating acceleration gave velocity and integrating velocity gave displacement and so no special differential equation techniques were required to solve the equations of motion.

In this unit situations where acceleration is dependent on displacement or velocity are considered and the technique of 'separation of variables' is required to solve the differential equations.

Since the motion being considered is in a straight line the scalar form of notation will be used for the calculus.

$$a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = \frac{dv}{dx}.v = v\frac{dv}{dx}$$

Which version of the acceleration is used in any situation depends on what is required and this is best illustrated in examples.

Worked examples

Example 1 (velocity as a function of displacement)
If $v = 2s^2 - 5s$ find (a) v when $s = 3$ m and (b) a when $s = 3$ m.

Solution

(a) $v = 2s^2 - 5s$, when $s = 3$; $v = 18 - 15 = 3$ ms^{-1}

(b) $a = v\dfrac{dv}{ds}$

$\dfrac{dv}{ds} = 4s - 5$

Hence, $a = v\dfrac{dv}{ds} = \left(2s^2 - 5s\right)\left(4s - 5\right)$

When $s = 3$, $a = 3 \times 7 = 21$ ms^{-2}

Example 2 (acceleration as a function of velocity)
A body of mass m moves in a straight line under the action of a force of magnitude $2\sqrt{v}$ per unit mass, acting in the direction of motion, where v is the speed of the body at time t.

If the body has initial speed u, find
(a) the time taken to reach a speed of $9u$, and
(b) the distance travelled in reaching this speed.

Solution

(a) $F = 2m\sqrt{v}$

$\Rightarrow ma = 2m\sqrt{v}$

$\Rightarrow a = 2v^{\frac{1}{2}}$ [$a = f(v)$ and **time** is required so use $a = \dfrac{dv}{dt}$]

$\Rightarrow \dfrac{dv}{dt} = 2v^{\frac{1}{2}}$

$\Rightarrow \displaystyle\int \frac{dv}{v^{\frac{1}{2}}} = \int 2dt$ [separating the variables]

$\Rightarrow \displaystyle\int v^{-\frac{1}{2}}dv = 2t + c$

$\Rightarrow 2v^{\frac{1}{2}} = 2t + c$

($v = u$ when $t = 0$ so $c = 2u^{\frac{1}{2}}$)

$\Rightarrow 2v^{\frac{1}{2}} = 2t + 2u^{\frac{1}{2}}$

$\Rightarrow t = \sqrt{v} - \sqrt{u}$

When $v = 9u$, $t = \sqrt{9u} - \sqrt{u} = 3\sqrt{u} - \sqrt{u} = 2\sqrt{u}$

(b) $a = 2v^{\frac{1}{2}}$

$\Rightarrow v\dfrac{dv}{dx} = 2v^{\frac{1}{2}}$ [$a = f(v)$ and **distance** required so we use $a = v\dfrac{dv}{dx}$]

$\Rightarrow \displaystyle\int_u^{9u} v^{\frac{1}{2}}dv = \int_0^x 2dx$ [separating the variables]

$\Rightarrow \left[\dfrac{2}{3}v^{\frac{3}{2}}\right]_u^{9u} = [2x]_0^x$

$\Rightarrow \dfrac{2}{3}\left[(9u)^{\frac{3}{2}} - \mathrm{u}^{\frac{3}{2}}\right] = 2x$

$\Rightarrow x = \dfrac{1}{3}\left(27u^{\frac{3}{2}} - u^{\frac{3}{2}}\right) = \dfrac{26}{3}u^{\frac{3}{2}}$

Note

A constant of integration was used in (a) and definite integration was used in (b) to illustrate that either technique can be used. Sometimes one is more convenient, sometimes the other.

Example 3 (acceleration as a function of displacement)

A body of mass 3 kg is projected from the origin with speed 4 ms^{-1} and moves in a straight line under the action of a force of $\dfrac{2}{\sqrt{x}}$ newtons per unit mass away from the origin, where x is the distance from the origin after time t seconds.

Find the distance travelled by the body in doubling its speed.

Solution

$$F = 3a$$
$$\Rightarrow 3a = \frac{6}{\sqrt{x}}$$
$$\Rightarrow v\frac{dv}{dx} = \frac{2}{x^{\frac{1}{2}}} \quad \text{[want to connect speed and distance so use } a = v\frac{dv}{dx}\text{]}$$
$$\Rightarrow \int_4^8 v\,dv = \int_0^x 2x^{-\frac{1}{2}}dx \quad \text{[separating the variables]}$$
$$\Rightarrow \left[\frac{v^2}{2}\right]_4^8 = \left[4x^{\frac{1}{2}}\right]_0^x$$
$$\Rightarrow \frac{64}{2} - \frac{16}{2} = 4\sqrt{x}$$
$$\Rightarrow 4\sqrt{x} = 24$$
$$\Rightarrow \sqrt{x} = 6$$
$$\Rightarrow x = 36$$

Thus the distance travelled by the body in doubling its speed is 36 metres.

The above examples have involved very basic integration but some more advanced integration may arise, such as in the following examples.

Example 4
A particle of unit mass, initially at rest at the origin, moves in a straight line under the action of a force of magnitude $30 - 2v$ newtons, where v ms^{-1} is the speed acquired in t seconds.

Find the time taken to reach a speed of 10 ms^{-1}.

Solution

$$a = 30 - 2v$$

$$\Rightarrow \frac{dv}{dt} = 30 - 2v = 2(15 - v) \quad \text{[wish to connect } v \text{ and } t \text{ so use } a = \frac{dv}{dt}\text{]}$$

$$\Rightarrow \int_0^{10} \frac{1}{15 - v}\,dv = \int_0^t 2dt \quad \text{[separating the variables]}$$

$$\Rightarrow \left[-\ln\left|15 - v\right|\right]_0^{10} = \left[2t\right]_0^t$$

$$\Rightarrow -\ln 5 + \ln 15 = 2t$$

$$\Rightarrow t = \frac{1}{2}\ln\frac{15}{5} = \frac{1}{2}\ln 3 = 0.55$$

Thus time required to reach a speed of 10 ms^{-1} is 0.55 s.

Terminal velocity

When a body is subject to a variable force such as that in the previous example, i.e. where the force decreases as the velocity increases, its velocity will increase until it reaches the value when its acceleration becomes zero. This is called its **terminal velocity** – the velocity a body approaches when moving in a resistive medium.

It is generally found in examples simply by equating the acceleration to zero. However, it can be illustrated mathematically in a different way.

Returning to the previous example:

$\frac{dv}{dt} = 2(15 - v) \Rightarrow a = 0$ when $v = 15$. Thus terminal speed is 15 ms^{-1}.

Alternatively, finding the speed acquired in time t:

$$\frac{dv}{dt} = 2(15 - v) \Rightarrow \int \frac{1}{15 - v}\,dv = \int 2dt \Rightarrow -\ln\left|15 - v\right| = 2t + c$$

Now, $v = 0$ when $t = 0$, so $c = -1\text{n}15$ and thus $2t = 1\text{n}15 - \ln\left|15 - v\right| = 1\text{n}\left|\frac{15}{15 - v}\right|$

Hence, $\frac{15}{15 - v} = e^{2t}$ [since $e^{2t} > 0$ for all real t]

Now, $\frac{15}{15 - v} = e^{2t}$

$$\Rightarrow 15 = 15e^{2t} - ve^{2t}$$

$$\Rightarrow v = \frac{15\left(e^{2t} - 1\right)}{e^{2t}}$$

$$\Rightarrow v = 15\left(1 - e^{-2t}\right)$$

As $t \to \infty$, $e^{-2t} \to 0$ and consequently $v \to 15$ and the terminal speed is 15 ms^{-1}.

Derivation of the equation $v^2 = \omega^2(a^2 - x^2)$ from the equation for simple harmonic motion

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\Rightarrow\ v\frac{dv}{dx} = -\omega^2 x$$

$$\Rightarrow\ \int v\,dv = -\int \omega^2 x\,dx \quad \text{[separating the variables]}$$

$$\Rightarrow\ \frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c$$

But $v = 0$ when $x = a$ so $c = \frac{1}{2}\omega^2 a^2$ (where a is the amplitude of the motio

Hence $\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega^2 a^2$ i.e. $v^2 = \omega^2(a^2 - x^2)$

Further worked examples

Example 5

A car of mass 1000 kg rolls from rest down a plane inclined at an angle $\sin^{-1}\left(\frac{1}{5}\right)$ to the horizontal. During the motion the car is subject to a resistance of magnitude $680 + 5v^2$ newtons when moving with speed v ms^{-1}. Write down an equation of motion for the car and find

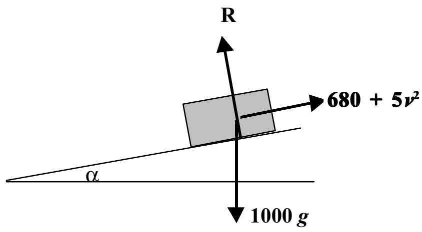
(a) the terminal speed of the car

(b) the distance travelled by the car in reaching a speed of 14 ms^{-1}

(c) the time taken to reach this speed.

[You may assume that $\dfrac{200}{256 - v^2} = \dfrac{25}{2}\left(\dfrac{1}{16 + v} + \dfrac{1}{16 - v}\right)$]

Solution



$$\alpha = \sin^{-1}\left(\frac{1}{5}\right)$$

Resolving perpendicular to the plane:

No motion in this direction, so $R = 1000g\cos\left(\sin^{-1}\frac{1}{5}\right)$

Resolving parallel to the plane:

Resultant force is $1000g\sin\left(\sin^{-1}\left(\frac{1}{5}\right)\right) - \left(680 + 5v^2\right)$

$$\text{i.e. } 1000g \times \left(\frac{1}{5}\right) - \left(680 + 5v^2\right)$$

$$\text{i.e. } 200 \times 9.8 - 680 - 5v^2$$

$$\text{i.e. } 1280 - 5v^2 \quad \text{[which diminishes as } v \text{ increases]}$$

Thus using '$F = ma$' we have: $1280 - 5v^2 = 1000a$

i.e. $256 - v^2 = 200a$

(a) The terminal speed occurs when $a = 0$

$a = 0 \Rightarrow v^2 = 256 \Rightarrow v = 16$

Thus terminal speed is 16 ms^{-1}.

(b)
$$256 - v^2 = 200a$$
$$\Rightarrow 256 - v^2 = 200v\frac{dv}{dx} \quad [\text{wish to connect } v \text{ and } x \text{ so use } a = v\frac{dv}{dx}]$$
$$\Rightarrow \frac{256 - v^2}{200v} = \frac{dv}{dx}$$
$$\Rightarrow \int_0^x 1dx = \int_0^{14} \frac{200v}{256 - v^2}dv \quad [\text{separating the variables}]$$
$$\Rightarrow x = -100\int_0^{14} \frac{-2v}{256 - v^2}dv$$
$$\Rightarrow x = -100\Big[\ln\left|256 - v^2\right|\Big]_0^{14} \quad [\text{using the substitution } u = 256 - v^2]$$
$$\Rightarrow x = -100(\ln 60 - \log 256) = 100\ln\frac{256}{60} = 145$$

Thus distance travelled to reach speed of 14 ms^{-1} is 145 metres.

(c)
$$256 - v^2 = 200a$$
$$\Rightarrow 256 - v^2 = 200\frac{dv}{dt} \quad [\text{wish to connect } v \text{ and } t \text{ so use } a = \frac{dv}{dt}]$$
$$\Rightarrow \int_0^t 1dt = \int_0^{14} \frac{200}{256 - v^2}dv \quad [\text{separating the variables}]$$
$$\Rightarrow t = \frac{25}{2} \times \int_0^{14}\left(\frac{1}{16 + v} + \frac{1}{16 - v}\right)dv$$
$$\Rightarrow t = \frac{25}{2}\Big[\ln|16 + v| - \ln|16 - v|\Big]_0^{14}$$
$$\Rightarrow t = \frac{25}{2}\left[\ln\left|\frac{16 + v}{16 - v}\right|\right]_0^{14}$$
$$\Rightarrow t = \frac{25}{2}(\ln 15 - \ln 1) = 33.85$$

Thus the time taken to reach speed of 14 ms^{-1} is approximately 34 seconds.

Example 6

A train of total mass 200 tonnes accelerates along a straight level stretch of track, against a resistance of 50 N per tonne.

Given that the train's engine is working at 500 kW, write down a differential equation for the motion and use it to find the distance travelled by the train in accelerating from 54 km per hour to 144 km per hour.

[You may assume that $\dfrac{20v^2}{50-v} = \dfrac{50\,000}{50-v} - 20v - 1000$.]

Solution

$$P = Fv \Rightarrow F = \frac{500\,000}{v}$$

$$200000a = F - R$$

$$\Rightarrow 200000a = \frac{500\,000}{v} - 200 \times 50$$

$$\Rightarrow 200\,000v\frac{dv}{dx} = \frac{500\,000}{v} - 10\,000$$

$$\Rightarrow 20v^2\frac{dv}{dx} = 50 - v$$

$$\Rightarrow \int_{15}^{40} \frac{20v^2}{50-v}dv = \int_0^x 1dx \text{ [separating variables: 54(144) km/h = 15(40) m/s]}$$

$$\Rightarrow x = \int_{15}^{40}\left(\frac{50000}{50-v} - 20v - 1000\right)dv$$

$$\Rightarrow x = \left[-50\,000\ln|50-v| - 10v^2 - 1000v\right]_{15}^{40}$$

$$\Rightarrow x = (-50\,000\ln 10 - 16\,000 - 40\,000) - (-50\,000\ln 35 - 2250 - 15\,000)$$

$$\Rightarrow x = 50\,000\ln 3.5 - 56\,000 + 17\,250 = 23888$$

Thus distance travelled in accelerating from 54 km/h to 144 km/h is approximately 23.9 km.

Note: These examples illustrate most of the integration techniques which generally arise, namely

- $\int \dfrac{1}{ax+b} = \dfrac{1}{a}\ln|ax+b| + c$
- $\int \dfrac{f'(x)}{f(x)}dx = \int \dfrac{1}{u}\,du = \ln u$ [using the substitution $u = f(x)$, $du = f'(x)dx$]
- use of **given** partial fractions.

Escape velocity

When a body is projected vertically from the surface of the Earth it is subject to the force of gravity and normally returns to the point of projection. However if the initial speed is great enough then the body will continue into space. The minimum velocity required to project a body into space is known as the escape velocity.

Suppose that a body of mass m is projected vertically upwards from a point O on the surface of the Earth with initial speed u. When the height of the body above O is r, then $m\ddot{r} = -\dfrac{mMG}{(R+r)^2}$

Now,
$$m\ddot{r} = -\frac{mMG}{(R+r)^2}$$
$$\Rightarrow \ddot{r} = -\frac{MG}{(R+r)^2}$$
$$\Rightarrow v\frac{dv}{dr} = -\frac{MG}{(R+r)^2}$$
$$\Rightarrow \int_u^v v\,dv = \int_0^r -\frac{MG}{(R+r)^2}\,dr$$
$$\Rightarrow \left[\frac{1}{2}v^2\right]_u^v = \left[\frac{MG}{(R+r)}\right]_0^r$$
$$\Rightarrow \frac{1}{2}(v^2-u^2) = \frac{MG}{(R+r)} - \frac{MG}{R} = -\frac{MGr}{(R+r)R}$$
$$\Rightarrow v^2 = u^2 - \frac{2MGr}{(R+r)R}$$

If the maximum height reached is H, then $v = 0$ when $r = H$.

Thus
$$0 = u^2 - \frac{2MGH}{(R+H)R}$$
$$\Rightarrow R^2u^2 + HRu^2 = 2MGH$$
$$\Rightarrow H = \frac{R^2u^2}{(2MG - Ru^2)}$$
$$\Rightarrow H = \frac{Ru^2}{\left(\frac{2MG}{R} - u^2\right)} \quad \left[\text{and } u^2 < \frac{2MG}{R} \text{ since } H > 0\right]$$

As $u^2 \to \dfrac{2MG}{R}$, $H \to \infty$ and so $u = \sqrt{\dfrac{2MG}{R}}$ is the escape speed.

Since $g = \dfrac{MG}{R^2}$, $u = \sqrt{2gR}$ and the escape speed (associated with the Earth) is 1.2×10^4 ms^{-1}

Resources/examples

S&T	Chapter 16 (Use of Calculus)		
		Pages 400–5	Ex 16B, Pages 405–6
		Pages 406–10	Ex 16C
		Pages 411–15	Ex 16D, Pages 416–18
		Pages 418–20	Ex 16E
RCS	Chapter 13 (Variable Acceleration)		
		Pages 249–51	Ex 13A, Page 252, Nos. 1–12
		Pages 253–6	Ex 13B, Page 257, Nos. 1–14
		Pages 258–60	Ex 13C, Nos. 1–7
		Pages 261–3	Ex 13D, Nos. 1–11
TG	Chapter 20 (Differential Equations in Mechanics)		
		Pages 357–61	Ex 20.1A, Page 363
			Ex 20.1B, Page 364
			Consolidation Ex.
B&C	Chapter 11 (General Motion of a Particle)		
		Pages 344–9	Ex 11c
		Pages 375–9	Miscellaneous Ex 11 (selected examples)
OG	No section on this work.		