Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.
   
   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics
3. Full credit will be given only where the solution contains appropriate working.
Section A (Mechanics 1 and 2)

Answer all the questions.

Candidates should observe that \( g \text{ m s}^{-2} \) denotes the magnitude of the acceleration due to gravity. Where appropriate, take its magnitude to be 9.8 m s\(^{-2}\).

A1. A particle has velocity \( 3t(2-t)\mathbf{j} \) where \( \mathbf{j} \) is the unit vector in the direction of motion. The time \( t \) is measured in seconds from the start of the motion and the displacement is measured in metres. Initially the particle is at the point with position vector \( 3\mathbf{j} \) relative to the origin \( O \). Calculate the distance of the particle from \( O \) when the velocity is a maximum.

A2. Towards the end of a long-distance race, Tessa is running at a uniform speed of 2 m s\(^{-1}\). When she is 120 m from the finishing line she starts to increase her speed. In doing so, she accelerates uniformly at 0.25 m s\(^{-2}\) for \( T \) seconds until she crosses the finishing line. Show that \( T \) satisfies the equation

\[ T^2 + 16T - 960 = 0 \]

and hence find her speed as she crosses the finishing line.

A3. A rough ramp \( OP \) of length 6 m is hinged at \( O \). A point \( P \) at the other end is able to move about \( O \) in a vertical plane as illustrated in the diagram. A small box of mass 2 kg is in equilibrium on the ramp.

(\( a \)) When \( P \) is 2 m above the horizontal plane through \( O \), the box is on the point of sliding down the ramp. Calculate the coefficient of friction between the box and the ramp.

(\( b \)) \( P \) is now raised to a height of 4 m above the horizontal plane. A force of \( F \) newtons, applied to the box and acting parallel to the ramp, is just sufficient to prevent the box from sliding down the ramp. Calculate the magnitude of \( F \).

A4. A bend on a smooth racing track forms an arc of a circle of radius \( r \) metres. The track is banked at an angle \( \alpha \) to the horizontal. A car takes the bend at speed \( v \text{ m s}^{-1} \) with no tendency to move either up or down the track. Express \( v \) in terms of \( \alpha, r \) and \( g \).
A5. Ben is cycling up a straight road which is inclined at an angle $\theta$ to the horizontal where $\sin \theta = \frac{1}{\sqrt{3}}$. The combined mass of Ben and the cycle is 100 kg. The resistance to the motion from non-gravitational forces is a force of magnitude $kv^2$ newtons, where $v \text{ m s}^{-1}$ is the speed of the cycle and $k$ is a constant.

When Ben is cycling up the road at 2 m s$^{-1}$, his acceleration is 0.05 m s$^{-2}$ and the rate at which he is working is 120 W. Calculate the value of the constant $k$.

A6. At 12 noon, an aircraft is above a point $A$ and is flying due West at a uniform speed of 180 km h$^{-1}$. Thirty minutes later, a second aircraft, which is flying at exactly the same height as the first with a uniform speed of 240 km h$^{-1}$, is 60 km due south of $A$. The aircraft are on a collision course.

(a) Calculate the time when the collision would take place if no evasive action were taken.

(b) Calculate the bearing on which the second aircraft is travelling.

A7. A parachutist with all her equipment has a total mass of 100 kg. She jumps from a helicopter which is hovering at a constant height and falls vertically under gravity. She experiences a resistive force of magnitude $(80 + 0.25v^2)$ newtons, where $v \text{ m s}^{-1}$ is her speed.

(a) Calculate the terminal velocity of the parachutist.

(b) Calculate the time for the parachutist to achieve 75% of her terminal velocity.

[You may use the result that $\int \frac{1}{a^2 - b^2 x^2} \, dx = \frac{1}{2ab} \ln \left(\frac{a + bx}{a - bx}\right)$]

A8. In a fairground game, a small target $T$ executes simple harmonic motion about a point $O$ with extreme points $A$ and $B$. When the target is 1 metre from $O$, its speed is $\frac{\pi}{\sqrt{3}}$ m s$^{-1}$ and when it is $\sqrt{3}$ metres from $O$ its speed is $\frac{\pi}{3}$ m s$^{-1}$.

(a) Show that the amplitude of the motion is 2 metres and calculate the period of the oscillation.

(b) A player has to shoot at the target, but it is only visible to the player when it is to the right of the point $P$ as shown in the diagram.

\[ A \quad \quad O \quad P \quad B \]

Given that the target takes 0.75 seconds to move from $P$ to $B$, calculate the distance $PB$. 

[Turn over]
A9. A ball of mass 0.1 kg is released from a point $A$ at the top of a smooth runway $AO$. The point $O$ is 1 metre above ground level and, when the ball reaches $O$, it falls to the ground under the action of gravity.

(a) Relative to the axes shown in Figure 1, the runway is modelled by the curve $y = \frac{1}{2}x^2$. The point $A$ is (–2, 2) and $O$ is the origin. The ball reaches the ground at $P(a, -1)$. Calculate the value of $a$.

(b) The track is modified to run between the same two points $A$ and $O$ with the shape modelled by the equation $y = x(x + 1)$ as shown in Figure 2.

Calculate the maximum height above ground level attained by the ball after it has passed through $O$. 

[Figure 1]

[Figure 2]
A10. Ice-dancers, Alice and Bob, are skating on a smooth ice rink.

Alice has mass $m\text{ kg}$ and is moving with a constant velocity $U\mathbf{i}$, where $\mathbf{i}$ is the unit vector in the direction of the $x$-axis and $U$ is measured in $\text{m s}^{-1}$.

Bob has mass $M\text{ kg}$ and is moving with a constant velocity $U\mathbf{j}$, where $\mathbf{j}$ is the unit vector in the direction of the $y$-axis.

The dancers collide and subsequently move off together with speed $V\text{ m s}^{-1}$ in a direction which makes an angle $\theta$ with the $x$-axis.

(a) Use conservation of momentum to show that

$$\tan \theta = \frac{M}{m}$$
and

$$V^2 = \frac{(m^2 + M^2)U^2}{(m + M)^2}.$$  

(b) Given that $\tan \theta = 2$, find an expression for the kinetic energy lost in the collision in terms of $m$ and $U$. 

[END OF SECTION A]

[Turn over for Section B on Pages six and seven]
Section B (Mathematics for Applied Mathematics)

Answer all the questions.

B1. Given that \(A, B, C\) and \(D\) are square matrices where:

\[
A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} x & 2 \\ 0 & y \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix}
\]

(a) Find \(AB\).  
(b) Express \(4C + D\) as a single matrix.  
(c) Given that \(AB = 4C + D\), find the values of \(x\) and \(y\).

B2. Given that \(y = e^{2x} \cos x\), find \(\frac{dy}{dx}\).

B3. Express \(y = \frac{4x-3}{x(x^2 + 3)}\), \(x \neq 0\), in partial fractions.

B4. (a) Use integration by parts to show that \(\int \ln x \, dx = x \ln x - x + c\).

(b) A goblet consists of a bowl and a short stem.  
The diagram below shows the bowl section of the goblet (on its side).  
The equation of the upper half of the curve is \(y = 2\sqrt{\ln x}\) for \(1 \leq x \leq 10\).

Given that the stem has length 1 and the overall height is 10, what is the capacity of the bowl?
B5.  \( \text{(a)} \) Use the standard formulas for \( \sum_{r=1}^{n} r \) and \( \sum_{r=1}^{n} r^2 \) to show that
\[
\sum_{r=1}^{n} (6r^2 - r) = \frac{1}{2} n(n+1)(4n+1).
\]

\( \text{(b)} \) Hence evaluate \( \sum_{r=5}^{10} (6r^2 - r) \).

\[
\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}.
\]

\[
B6. \text{ Newton’s law of cooling states that a body loses heat at a rate which is proportional to the difference in temperature between itself and its surroundings. So, in a room with constant temperature 22 °C, the temperature } T \degree \text{C of a body after a time } t \text{ minutes satisfies}
\]
\[
\frac{dT}{dt} = k(T - 22)
\]
where \( k \) is a negative constant.

\( \text{(a)} \) Hence show that \( T \) can be expressed in the form \( T = Ae^{kt} + 22 \) for some arbitrary constant \( A \).

\( \text{(b)} \) In a restaurant, where the temperature remains constant at 22 °C, a freshly baked roll, with temperature 82 °C, is placed on a cooling tray. After 5 minutes, the temperature of the roll has fallen by 20 degrees. Calculate the values of \( A \) and \( k \).

Write down an expression for the temperature of the roll after \( t \) minutes. Supposing the roll remains uneaten after a further 5 minutes, what will its temperature be?