Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.
   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics
3. Full credit will be given only where the solution contains appropriate working.
Section A (Mechanics 1 and 2)

Answer all the questions.

Candidates should observe that $g \text{ m s}^{-2}$ denotes the magnitude of the acceleration due to gravity.

Where appropriate, take its magnitude to be $9.8 \text{ m s}^{-2}$.

A1. Relative to a rectangular coordinate system, the position of an ice skater at time $t$ seconds is

$$\mathbf{r}(t) = \left( \frac{1}{3} t^3 - 4t^2 \right) \mathbf{i} - (2t^2 - 1) \mathbf{j},$$

where $\mathbf{i}$, $\mathbf{j}$ are the unit vectors in the $x$, $y$ directions respectively and distances are measured in metres.

Find the speed of the ice skater at the instant when the acceleration is parallel to the $y$-axis.

A2. A piston oscillates about the point $O$ with simple harmonic motion of amplitude $0.25 \text{ m}$.

Calculate the distance of the piston from $O$ when its speed is half its maximum speed.

A3. A lift is initially at rest at ground level. It begins to accelerate upwards at $\frac{1}{8} g \text{ m s}^{-2}$. At the same instant, a light bulb in the ceiling of the lift begins to fall towards the lift floor. The initial distance between the lift floor and the light bulb is 2 metres.

(a) Measuring distances in metres relative to the ground level, show that the position of the light bulb relative to the lift floor is

$$\left(2 - \frac{9}{16} gt^2\right) \mathbf{j},$$

where $\mathbf{j}$ is the unit vector in the upward vertical direction, and $t$ is the time in seconds from the start of the motion of the lift.

(b) Calculate the distance the light bulb falls before hitting the lift floor.
A4. A golfer strikes a golf ball from $O$ across a horizontal section of ground, giving the ball an initial speed of $V \text{ m s}^{-1}$ at an angle $\alpha$ to the horizontal.

(a) Show that the range, $R$ metres, of the golf ball is given by

$$R = \frac{V^2}{g} \sin 2\alpha.$$  

(b) The golfer intends the ball to land between two points $A$ and $B$ on the horizontal section such that $OA = L$ metres, $OB = 2L$ metres and $OAB$ is a straight line.

Given that the angle of projection of the ball is $15^\circ$, show that the initial speed must satisfy

$$\sqrt{2} < \frac{V}{\sqrt{gL}} < 2.$$ 

A5. A railway truck of mass $3m$ kilograms travelling at $u \text{ m s}^{-1}$ along a straight horizontal track, collides and couples with a stationary truck of mass $m$ kilograms. Due to the action of a constant resistive force of magnitude $R$ newtons, the two trucks come to rest $T$ seconds after the collision.

(a) Determine an expression for $R$ in terms of $m$, $u$ and $T$.

(b) Find an expression, in terms of $m$ and $u$, for the work done by $R$ in bringing the trucks to rest.

A6. A conical pendulum consists of a bobbin of mass $m$ kilograms attached to one end, $B$, of a light elastic string $AB$ of natural length $l$ metres and modulus of elasticity $8mg$ newtons. The other end, $A$, of the string is held fixed. The bobbin moves in a horizontal circle with centre vertically below $A$, such that the angle between the string $AB$ and the vertical is $45^\circ$.

(a) Determine, in terms of $l$, the extension of the string beyond its natural length.

(b) Show that the angular speed, $\omega$ radians per second, of the bobbin is given by

$$\omega^2 = \frac{8g}{(1 + 4\sqrt{2})l}.$$ 

[Turn over]
A7. Alan pulls a container with weight of magnitude $W$ newtons at a constant speed up a rough plane, with coefficient of friction $\mu$, inclined at an acute angle $\theta$ to the horizontal by means of a light inextensible rope, as shown below. The rope also makes an angle $\theta$ to the inclined plane.

(a) Show that the magnitude of the tension in the rope is given by

$$\left(\frac{\tan \theta + \mu}{1 + \mu \tan \theta}\right) W$$

newtons.

(b) Determine the range of values of $\theta$ for which the tension in the rope is less than the weight of the container.

A8. A mass $m$ kilograms is attached to one end, $A$, of a light inextensible string of length $L$ metres, the other end of which is fixed at a point $O$. Initially the mass hangs vertically below $O$ with the string taut. The mass is then given a horizontal speed of $\sqrt{\frac{g}{2}}L$ ms$^{-1}$, causing it to start to travel in a vertical circle of centre $O$. Subsequently, the string $OA$ makes an angle $\theta$ with the downward vertical through $O$.

(a) When $\theta = 45^\circ$, find expressions for:

(i) the speed of the mass in terms of $L$ and $g$;
(ii) the magnitude of the tension in the string, in terms of $m$ and $g$.

(b) Determine the value of $\theta$ at which the string first becomes slack.
A9. A box of mass \( m \) kilograms is dropped from rest from a point \( A \) at a height \( h \) metres above ground level. The box experiences a force of magnitude \( mkv^2 \) newtons due to air resistance, where \( v \) m s\(^{-1}\) is the speed of the box and \( k \) is a constant.

(a) Show that the speed of the box satisfies the differential equation

\[
v \frac{dv}{dx} = g - kv^2,
\]

where \( x \) metres is the distance fallen by the box from \( A \).

(b) By making the substitution \( w = g - kv^2 \), or otherwise, solve the differential equation in (a) to show that

\[
v^2 = \frac{g}{k}(1 - e^{-2kv}).
\]

(c) Given that \( kh = 2 \), calculate what fraction of the initial potential energy is used in doing work against the resistive force during the descent of the box to the ground.
Section B (Mathematics for Applied Mathematics)

Answer all the questions.

B1. Calculate \( A^{-1} \) where
\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}
\]

Hence solve the system of equations
\[
\begin{align*}
x + y &= 1 \\
2x + 3y + z &= 2 \\
2x + 2y + z &= 1.
\end{align*}
\]

B2. Given that \( y = \ln(1 + \sin x) \), where \( 0 < x < \frac{\pi}{2} \), show that \( \frac{d^2 y}{dx^2} = \frac{-1}{1 + \sin x} \).

B3. Define \( S_n = \sum_{r=1}^{n} r^2, \ n \geq 1 \). Write down formulae for \( S_n \) and \( S_{2n+1} \).

Obtain a formula for \( 2^2 + 4^2 + \ldots + (2n)^2 \).

B4. Solve the differential equation
\[
\cos^2 x \frac{dy}{dx} = y,
\]
given that \( y > 0 \) and that \( y = 2 \) when \( x = 0 \).

B5. Use the substitution \( 1 + x^2 = u \) to obtain
\[
\int \frac{x^3}{\sqrt{1 + x^2}} \, dx.
\]

B6. (a) Evaluate \( \int_{0}^{1} xe^{2x} \, dx \).

(b) Use part (a) to evaluate \( \int_{0}^{1} x^2 e^{2x} \, dx \).

(c) Hence obtain \( \int_{0}^{1} (3x^2 + 2x)e^{2x} \, dx \).

[END OF SECTION B]

[END OF QUESTION PAPER]