2004 Mathematics

Advanced Higher

Finalised Marking Instructions
Solutions to Advanced Higher Mathematics Paper

1. (a) \[ f(x) = \cos^2 x e^{\tan x} \]
\[ f'(x) = 2(-\sin x) \cos x e^{\tan x} + \cos^2 x \sec^2 x e^{\tan x} \]
1 for Product Rule
2 for accuracy
\[ = (1 - \sin 2x) e^{\tan x} \]
\[ f'\left(\frac{\pi}{4}\right) = (1 - \sin \frac{\pi}{2}) e^{\tan \pi/4} = 0. \]

(b) \[ g(x) = \frac{\tan^{-1} 2x}{1 + 4x^2} \]
\[ g'(x) = \frac{\frac{2}{1 + 4x^2}(1 + 4x^2) - \tan^{-1} 2x (8x)}{(1 + 4x^2)^2} \]
1 for Product Rule
2 for accuracy
\[ = \frac{2 - 8x \tan^{-1} 2x}{(1 + 4x^2)^2} \]

2. \[(a^2 - 3)^4 = (a^2)^4 + 4(a^2)^3 (-3) + 6(a^2)^2 (-3)^2 + 4(a^2)(-3)^3 + (-3)^4 \]
1 for binomial coefficients
\[ = a^8 - 12a^6 + 54a^4 - 108a^2 + 81 \]
1 for powers
1 for coefficients

3. \[ x = 5 \cos \theta \Rightarrow \frac{dx}{d\theta} = -5 \sin \theta \]
\[ y = 5 \sin \theta \Rightarrow \frac{dy}{d\theta} = 5 \cos \theta \]
1
\[ \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5 \cos \theta}{-5 \sin \theta} \]
1
When \( \theta = \frac{\pi}{4}, \frac{dy}{dx} = \frac{-1}{\sqrt{2}} = -1, \)
1
\[ x = \frac{5}{\sqrt{2}}, y = \frac{5}{\sqrt{2}} \]
so an equation of the tangent is
\[ y - \frac{5}{\sqrt{2}} = -\left(x - \frac{5}{\sqrt{2}}\right) \]
1
i.e. \( x + y = 5\sqrt{2}. \)
4. 

\[ z^2(z + 3) = (1 + 4i - 4)(1 + 2i + 3) \]

\[ = (-3 + 4i)(4 + 2i) \]

\[ = -20 + 10i \]

\[ z^3 + 3z^2 - 5z + 25 = z^2(z + 3) - 5z + 25 \]

\[ = -20 + 10i - 5 - 10i + 25 = 0 \]

Note: direct substitution of \(1 + 2i\) into \(z^3 + 3z^2 - 5z + 25\) was acceptable.

Another root is the conjugate of \(z\), i.e. \(1 - 2i\).

The corresponding quadratic factor is \((z - 1)^2 + 4 = z^2 - 2z + 5\).

\[ z^3 + 3z^2 - 5z + 25 = (z - 2z + 5)(z + 5) \]

\[ z = -5 \]

Note: any valid method was acceptable.

5. 

\[ \frac{1}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2} \]

\[ = \frac{1}{5(x - 3)} - \frac{1}{5(x + 2)} \]

\[ \int_0^1 \frac{1}{x^2 - x - 6} \, dx = \frac{1}{5} \int_0^1 \frac{1}{|x - 3|} - \frac{1}{|x + 2|} \, dx \]

\[ = \frac{1}{5} \left[ \ln |x - 3| - \ln |x + 2| \right]_0^1 \]

\[ = \frac{1}{5} \left[ \ln \frac{2}{3} - \ln \frac{3}{2} \right] \]

\[ = \frac{1}{5} \ln \frac{4}{9} \approx -0.162 \]

6. 

\[ M_1 = \begin{pmatrix} \cos \frac{\pi}{2} - \sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} \cos \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ M_2M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \]

The transformation represented by \(M_2M_1\) is reflection in \(y = -x\).
7. \[ f(x) = e^x \sin x \quad f(0) = 0 \]
\[ f'(x) = e^x \sin x + e^x \cos x \quad f'(0) = 1 \]
\[ f''(x) = e^x \sin x + e^x \cos x - e^x \sin x + e^x \cos x \quad f''(0) = 2 \]
\[ = 2e^x \cos x \]
\[ f'''(x) = 2e^x \cos x - 2e^x \sin x \quad f'''(0) = 2 \]
\[ f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots \]
\[ e^x \sin x = x + x^2 + \frac{1}{3!}x^3 - \ldots \]

OR
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]
\[ \sin x = x - \frac{x^3}{3!} + \ldots \]
\[ e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) \quad \text{1 – method} \]
\[ = x - \frac{x^3}{6} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \ldots \]
\[ = x + x^2 + \frac{x^3}{3} - \ldots \]

8. \[ 231 = 13 \times 17 + 10 \quad \text{1 for method} \]
\[ 17 = 1 \times 10 + 7 \]
\[ 10 = 1 \times 7 + 3 \]
\[ 7 = 2 \times 3 + 1 \]

Thus the highest common factor is 1.
\[ 1 = 7 - 2 \times 3 \]
\[ = 7 - 2 \times (10 - 7) = 3 \times 7 - 2 \times 10 \quad \text{1 for method} \]
\[ = 3 \times (17 - 10) - 2 \times 10 = 3 \times 17 - 5 \times 10 \]
\[ = 3 \times 17 - 5 \times (231 - 13 \times 17) = 68 \times 17 - 5 \times 231. \]

So \( x = -5 \) and \( y = 68 \).

9. \[ x = (u - 1)^2 \Rightarrow dx = 2(u - 1)du \]
\[ \int \frac{1}{(1 + \sqrt{x})} dx = \int \frac{2(u - 1)}{u^3} du \]
\[ = 2 \int (u^2 - u^{-3}) du \]
\[ = 2\left(\frac{-1}{u} + \frac{1}{2u^2}\right) + c \]
\[ = \left(\frac{1}{(1 + \sqrt{x})^2} - \frac{2}{(1 + \sqrt{x})}\right) + c \]
10. \( f(x) = x^4 \sin 2x \) so

\[
f(-x) = (-x)^4 \sin (-2x) = -x^4 \sin 2x = -f(x)
\]

So \( f(x) = x^4 \sin 2x \) is an odd function.

Note: a sketch given with a comment and correct answer, give full marks.
A sketch without a comment, gets a maximum of two marks.

11. \[
V = \int_a^b \pi y^2 \, dx
\]

\[
= \pi \int_0^1 e^{-4x} \, dx
\]

\[
= \pi \left[ -\frac{e^{-4x}}{4} \right]_0^1
\]

\[
= \pi \left[ -\frac{1}{4e^4} + \frac{1}{4} \right]
\]

\[
= \frac{\pi}{4} \left[ 1 - \frac{1}{e^4} \right] = 0.7706
\]

12. \[
LHS = \frac{d}{dx}(xe^x) = xe^x + 1e^x = (x + 1)e^x
\]

\[
RHS = (x + 1)e^x
\]

So true when \( n = 1 \).

Assume \( \frac{d^k}{dx^k}(xe^x) = (x + k)e^x \)

Consider \[
\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx} \left( \frac{d^k}{dx^k}(xe^x) \right)
\]

\[
= \frac{d}{dx}((x + k)e^x) = e^x + (x + k)e^x
\]

So true for \( k \) means it is true for \( (k + 1) \), therefore it is true for all integers \( n \geq 1 \).
13. (a) \[ y = \frac{x - 3}{x + 2} = 1 - \frac{5}{x + 2} \]
Vertical asymptote is \( x = -2 \).
Horizontal asymptote is \( y = 1 \).

(b) \[ \frac{dy}{dx} = \frac{5}{(x + 2)^2} \]
\( \neq 0 \)

(c) \[ \frac{d^2y}{dx^2} = \frac{-10}{(x + 2)^3} \neq 0 \]
So there are no points of inflexion.

(d) 
\[ \text{The asymptotes are } x = 1 \text{ and } y = -2. \]
The domain must exclude \( x = 1 \).

*Note: candidates are not required to obtain a formula for } f^{-1}.

14. (a) \[ \overrightarrow{AB} = -i + 2j - 4k, \overrightarrow{AC} = 0i + j - 3k \]
\[ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & -4 \\ 0 & 1 & -3 \end{vmatrix} = -2i - 3j - k \]
\( 1 \text{ for method} \)
\( 1 \text{ for accuracy} \)
\[-2x - 3y - z = c (= -2 + 0 - 3 = -5) \]
i.e. an equation for \( \pi_1 \) is \( 2x + 3y + z = 5 \).

Let an angle be \( \theta \), then
\[ \cos \theta = \frac{(2i + 3j + k) \cdot (i + j - k)}{\sqrt{4 + 9 + 1} \sqrt{1 + 1 + 1}} \]
\[ = \frac{2 + 3 - 1}{\sqrt{14} \times 3} \]
\[ = \frac{4}{\sqrt{42}} \]
\[ \theta = 51.9^\circ \]

*Note: an acute angle is required.
(b) Let \( \frac{x - 11}{4} = \frac{y - 15}{5} = \frac{z - 12}{2} = t. \)

Then \( x = 4t + 11; y = 5t + 15; z = 2t + 12 \)

\[
(4t + 11) + (5t + 15) - (2t + 12) = 0
\]

\[7t = -14 \implies t = -2 \]

\[x = 3; y = 5 \text{ and } z = 8.\]

15. (a) \[
\frac{dy}{dx} - 3y = x^4
\]

\[
\frac{dy}{dx} - \frac{3}{x} y = x^3
\]

Integrating factor is 
\[
e^{-\int \frac{3}{x} dx} = x^{-3}.
\]

\[
\frac{1}{x^3} \frac{dy}{dx} - \frac{3}{x^4} y = 1
\]

\[
\frac{d}{dx} \left( \frac{1}{x^3} y \right) = 1
\]

\[
\frac{y}{x^3} = x + c
\]

\[
y = (x + c)x^3
\]

\[y = 2 \text{ when } x = 1, \text{ so } 2 = 1 + c \]

\[c = 1
\]

\[y = (x + 1)x^3
\]

(b) \[
y \frac{dy}{dx} - 3x = x^4
\]

\[
y \frac{dy}{dx} = x^4 + 3x
\]

\[
\int y \, dy = \int (x^4 + 3x) \, dx
\]

\[
\frac{y^2}{2} = \frac{x^5}{5} + \frac{3x^2}{2} + c'
\]

When \( x = 1, y = 2 \) so \( c' = 2 - \frac{1}{5} - \frac{3}{2} = \frac{3}{10} \) and so 

\[
y = \sqrt{2 \left( \frac{x^5}{5} + \frac{3x^2}{2} + \frac{3}{10} \right)}.
\]
16. (a) The series is arithmetic with \( a = 8, d = 3 \) and \( n = 17 \).
\[
S = \frac{n}{2} \left\{ 2a + (n - 1)d \right\} = \frac{17}{2} \left\{ 16 + 16 \times 3 \right\} = 17 \times 32 = 544
\]

(b) \( a = 2, S_3 = a + ar + ar^2 = 266 \). Since \( a = 2 \)
\[
r^2 + r + 1 = 133
\]
\[
r^2 + r - 132 = 0
\]
\[
(r - 11)(r + 12) = 0
\]
\[
r = 11 \) (since terms are positive).
\[
\text{Note: other valid equations could be used.}
\]

(c) \[
2 (2a + 3 \times 2) = a \left( 1 + 2 + 2^2 + 2^3 \right)
\]
\[
4a + 12 = 15a
\]
\[
11a = 12
\]
\[
a = \frac{12}{11}
\]

The sum \( S_B = \frac{12}{11} (2^n - 1) \) and \( S_A = \frac{a}{7} \left( \frac{2^4}{7} + 2(n - 1) \right) = n \left( \frac{1}{4} + n \right) \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
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<tbody>
<tr>
<td>( S_B )</td>
<td>[\frac{180}{11}]</td>
<td>[\frac{232}{11}]</td>
<td>[\frac{266}{11}]</td>
<td>[\frac{1524}{11}]</td>
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<tr>
<td>( S_A )</td>
<td>[\frac{180}{11}]</td>
<td>[\frac{280}{11}]</td>
<td>[\frac{402}{11}]</td>
<td>[\frac{546}{11}]</td>
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</table>

The smallest \( n \) is 7.
2004 Applied Mathematics

Advanced Higher – Section A

Finalised Marking Instructions
Advanced Higher Applied Mathematics 2004
Solutions for Section A (Statistics 1 and 2)

A1. (a) Stratified
and Quota [or Quota (convenience)]
(b) Approach (a) should be best:
since (b) is not random (other forms e.g. Glasgow not typical, biased)

A2. (a) $F - Bin(192, 0.002)$.
(b) $P(F \geq 3) = 1 - P(F < 2)$
    $= 1 - (0.6809 + 0.2620 + 0.0501)$
    $= 0.0070$

Notes: applying a Poisson distribution loses (at least) one mark; a Normal
distribution loses two marks.
(c) Approximate using the Foi(0.384)

A3. Assume that yields are normally distributed.
[Random or independent will not do.]
$\bar{x} = 404.2; \sigma = 10.03$
$t = 2.776$
A 95% confidence interval for the mean yield, $\mu$, is given by:-
$\bar{x} \pm t \frac{\sigma}{\sqrt{n}}$
$= 404.2 \pm 2.776 \frac{10.03}{\sqrt{5}}$
$= 404.2 \pm 12.45$
or (391.75, 416.65).
The fact that the confidence interval does not include 382
provides evidence, at the 5% level of significance, of a
change in the mean yield. (Stating it is changed loses one mark.)
Note: the third and fourth marks are lost if a z interval is used.

A4. TNE = 3% of 500 = 15
With maximum allowable standard deviation
$P(\text{weight} < 485) = 0.025$
$\Rightarrow \frac{485 - 505}{\sigma} = -1.96$
$\Rightarrow \sigma = \frac{20}{1.96} = 10.2$

There will be a small probability of obtaining a content
weight less than 470g with the normal model.
A5. Assume that the distributions of times Before and After have the same shape.  

Notes: a Normal distribution with the same shape is a valid comment. 
Independent, random, Normal (without shape) are not valid.  
Null hypothesis $H_0$: Median After = Median Before  
Alternative hypothesis $H_1$: Median After < Median Before

<table>
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<tr>
<th>Time</th>
<th>19</th>
<th>29</th>
<th>31</th>
<th>35</th>
<th>37</th>
<th>39</th>
<th>39</th>
<th>41</th>
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<th>45</th>
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<td>B</td>
<td>B</td>
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<td>B</td>
<td>B</td>
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<tr>
<td>Rank</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6.5</td>
<td>6.5</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
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Rank sum for After times = 37.5  
$W - \frac{1}{2}n(n + 1) = 37.5 - 28 = 9.5$  
$P(W - \frac{1}{2}n(n + 1) < 10)$  
$= \frac{125}{3432}$  
$= 0.036$  

Since this value is less than 0.05 the null hypothesis would be rejected in favour of the alternative, indicating evidence of improved performance.  

Notes:  
As the computed value, 9.5, is not in the tables, a range of values for the probability was acceptable.  
A Normal approximation was accepted.

A6. 

<table>
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<th>Cream</th>
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<th>B</th>
<th>C</th>
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<tr>
<td>Obs. No. of purchasers</td>
<td>66</td>
<td>99</td>
<td>75</td>
</tr>
<tr>
<td>Exp. No. of purchasers</td>
<td>80</td>
<td>80</td>
<td>80</td>
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</table>

$X^2 = \sum \frac{(O - E)^2}{E}$  
$= \frac{(66 - 80)^2}{80} + \frac{(99 - 80)^2}{80} + \frac{(75 - 80)^2}{80}$  
$= 2.45 + 4.5125 + 0.3125 = 7.275$  
with 2 d.f.  
The critical value of chi-squared at the 5% level is 5.991  
so the null hypothesis would be rejected.  
i.e. there is evidence of a preference.  
The fact that the p-value is less than 0.05 confirms rejection of the null hypothesis at the 5% level of significance.  
Note: using a two-tail test loses a mark.
A7. (a) The fitted value is 13.791 with residual 10.209.
(b) The wedge-shaped plot casts doubt on the assumption of constant variance of $Y_i$ (i.e. variance not constant) 1
(c) Satisfactory now since variance seems to be more constant. 
*Note: A phrase such as 'more randomly scattered' is acceptable.*
(d) The residuals are normally distributed.

A8. (a) 

<table>
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</table>

Assume that differences are independent.

$H_0$: Median (Post - Pre) = 0 [or $\eta_d = 0$]

$H_1$: Median (Post - Pre) > 0 [or $\eta_d > 0$]

Under $H_0$ the differences Bin (11,0.5) with $b = 2$.

$P(B < 2) = (C_0^{11} + C_1^{11} + C_2^{11})0.5^{11}$

$= (1 + 11 + 55)0.5^{11} = 0.0327$. 1

Since 0.0327 < 0.05 the null hypothesis is rejected and there is evidence that the median PCS-12 score has gone up.

*Note: applying a two-tailed test loses a mark.*

(b) $H_0 : \mu_{Post} = 50$

$H_1 : \mu_{Post} \neq 50$

$\bar{x} = 48.42$

$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{48.42 - 50}{10/\sqrt{12}} = -0.55$. 1,1

The critical region is $|z| > 1.96$ at the 5% level of significance.

Since -0.55 is not in the critical region, the null hypothesis is accepted indicating that the Post-operation scores are consistent with a population mean of 50.

*Note: a correct use of probability comparisons gets full marks.*
A9. (a) \[ P(\text{Alaskan fish classified as Canadian}) \]
\[ = P(X > 120 \mid X \sim N(100, 20^2)) \]
\[ = P\left(Z > \frac{120 - 100}{20}\right) \]
\[ = P(Z > 1) \]
\[ = 0.1587 \]

(b) The probability is the same as in (a) because of symmetry.

(c) \[ P(\text{Canadian origin} \mid \text{Alaskan predicted}) \]
\[ = \frac{P(\text{Alaskan predicted and Canadian origin})}{P(\text{Alaskan predicted})} \]
\[ = \frac{P(\text{Ala pred and Alaskan}) + P(\text{Ala pred but Canadian})}{0.4 \times 0.1587} \]
\[ = \frac{0.6 \times 0.8413 + 0.4 \times 0.1587}{0.06348} \]
\[ = \frac{0.50478 + 0.06348}{0.112}. \]

Note: Alternative methods acceptable e.g. Venn or Tree Diagrams

A10. The number, \(X\), of inaccurate invoices in samples of \(n\) will have the Binomial \((n, p)\) distribution so

\[ V(X) = npq \]
\[ = np(1 - p) \]

\[ \Rightarrow V(\text{Proportion}) = V\left(\frac{1}{n} X\right) = \frac{1}{n^2} V(X) \]
\[ = \frac{p(1 - p)}{n} \]

\[ \Rightarrow \text{Standard deviation of Proportion} = \sqrt{\frac{p(1 - p)}{n}}. \]

(a) UCL = \(p + 3\sqrt{\frac{p(1 - p)}{n}}\)
\[ = 0.12 + 3\sqrt{\frac{0.12 \times 0.88}{150}} \]
\[ = 0.12 + 0.08 = 0.20. \]

LCL = 0.12 - 0.08 = 0.04

(b) The fact that the point for Week 30 falls below the lower chart limit provides evidence of a drop in the proportion of inaccurate invoices. Or: 8 consecutive points fell below the centre line.

(c) A new chart should be constructed (or set new limits) using an estimate of \(p\) for calculation of limits which is based on data collected since the process change.

\[ page 5 \]
2004 Applied Mathematics

Advanced Higher – Section B

Finalised Marking Instructions
Advanced Higher Applied Mathematics 2004  
Solutions for Section B (Numerical Analysis 1 and 2)

B1. \[ f(x) = \ln(2-x) \quad f'(x) = -\frac{1}{(2-x)} \quad f''(x) = \frac{-1}{(2-x)^2} \quad f'''(x) = \frac{-2}{(2-x)^3} \]

Taylor polynomial is

\[ p(1+h) = \ln 1 - h - \frac{h^2}{2} - \frac{h^3}{6} \]

For \( \ln 1.1, h = 0.1 \) and \( p(0.9) = 0.1 - 0.005 + 0.00033 = 0.0953 \).

Hence expect \( f(x) \) to be more sensitive in \( l_2 \) since coefficient of \( h \) is much larger.

B2. \[ L(2.5) \]

\[ \begin{align*}
&= \frac{(2.5-1.5)(2.5-3.0)(2.5-4.5)}{(0.5-1.5)(0.5-3.0)(0.5-4.5)} \cdot 1.737 + \frac{(2.5-0.5)(2.5-3.0)(2.5-4.5)}{(1.5-0.5)(1.5-3.0)(1.5-4.5)} \cdot 2.412 \\
&\quad + \frac{(2.5-0.5)(2.5-1.5)(2.5-4.5)}{(3.0-0.5)(3.0-1.5)(3.0-4.5)} \cdot 3.284 + \frac{(2.5-0.5)(2.5-1.5)(2.5-3.0)}{(4.5-0.5)(4.5-1.5)(4.5-3.0)} \cdot 2.797 \\
&= -\frac{1 \times 1.737}{10} + \frac{2 \times 2.412}{4.5} + \frac{4 \times 3.284}{2.5 \times 2.25} - \frac{1 \times 2.797}{18} \\
&= -0.1737 + 0.4694 + 2.3695 - 0.1554 = 3.078
\]

B3. \[ \Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0 \]

Maximum rounding error = \( \epsilon + 2\epsilon + \epsilon = 4\epsilon \).

\[ \Delta^2 f_0 = 2.618 - 2 \times 2.369 + 2.124 = 0.004 \]

and \( 4\epsilon = 4 \times 0.0005 = 0.002 \).

\( \Delta^2 f_0 \) appears to be significantly different from 0.
B4. (a) Difference table is:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x$</th>
<th>$f(x)$</th>
<th>diff1</th>
<th>diff2</th>
<th>diff3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.023</td>
<td>352</td>
<td>-95</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>1.375</td>
<td>257</td>
<td>-92</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.632</td>
<td>165</td>
<td>-96</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>1.797</td>
<td>69</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1.866</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) \( p = 0.3 \)

\[
f'(0.65) = 1.375 + 0.3(0.257) + \frac{(0.3)(-0.7)}{2} (-0.092)
= 1.375 + 0.077 + 0.010 = 1.462
\]

(or, with \( p = 1.3, 1.023 + 0.458 - 0.019 \)).

B5. \( f(x) = (((x - 1.1)x + 1.7)x)x - 3.2 \) and \( f(1.3) = 0.1124 \).
Since \( f \) is positive and increasing at \( x = 1.3 \), root appears to occur for \( x < 1.3 \).
\[
f'(x)_{\text{min}} = (((x - 1.15)x + 1.65)x)x - 3.25
f'(1.3)_{\text{min}} = -0.132 \text{ (opposite sign), so root may occur for } x > 1.3.
\]

B6. In diagonally dominant form,

\[
4x_1 - 0.3x_2 + 0.5x_3 = 6.1
0.5x_1 - 7x_2 + 0.7x_3 = 3.7
0.3x_1 + 2x_3 = 8.6.
\]

The diagonal coefficients of \( x \) are large relative to the others, so system is likely to be stable. (Or, this implies equations are highly linearly independent, or, determinant of system is large.)

Rewritten equations are:

\[
x_1 = (6.1 + 0.3x_2 - 0.5x_3) / 4
x_2 = (-3.7 + 0.5x_2 + 0.7x_3) / 7
x_3 = (8.6 - 0.3x_1) / 2
\]

Gauss Seidel table is:

\[
\begin{array}{ccc}
x_1 & x_2 & x_3 \\
0 & 0 & 0 \\
1.525 & -0.420 & 4.071 \\
0.985 & -0.051 & 4.152 \\
1.002 & -0.042 & 4.150 \\
1.003 & -0.042 & \\
\end{array}
\]

Hence (2 decimal places) \( x_1 = 1.00; \ x_2 = -0.04; \ x_3 = 4.15. \)
B7. Tableau is:
\[
\begin{pmatrix}
2.6 & 0 & 1.622 & 0.742 & 0.479 & 0 \\
0 & 6.469 & 1.923 & -0.538 & 1 & 0 \\
0 & 0 & 3.604 & -0.415 & 0.128 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
2.6 & 0 & 0 & 0.929 & 0.421 & -0.450 \\
0 & 6.469 & 0 & -0.317 & 0.932 & -0.534 \\
0 & 0 & 3.604 & -0.415 & 0.128 & 1
\end{pmatrix}
\]
\[
(R_1 - 1.622 R_2 / 3.604) \\
(R_2 - 1.923 R_3 / 3.604)
\]
\[
\begin{pmatrix}
1 & 0 & 0 & 0.357 & 0.162 & -0.173 \\
0 & 1 & 0 & -0.049 & 0.144 & -0.083 \\
0 & 0 & 1 & -0.115 & 0.036 & 0.277
\end{pmatrix}
\]
(dividing by diagonal elements)

Hence \( A^{-1} = \begin{pmatrix}
0.36 & 0.16 & -0.17 \\
-0.05 & 0.14 & -0.08 \\
-0.12 & 0.04 & 0.28
\end{pmatrix} \) accuracy 1

B8. (a)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( f(x, y) )</th>
<th>( hf(x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.414</td>
<td>0.041</td>
</tr>
<tr>
<td>1.1</td>
<td>1.041</td>
<td>0.514</td>
<td>0.051</td>
</tr>
<tr>
<td>1.2</td>
<td>1.092</td>
<td>0.620</td>
<td>0.062</td>
</tr>
<tr>
<td>1.3</td>
<td>1.154</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Global truncation error is first order.

(b) Predictor-corrector calculation (with one corrector application) is:

\[
x \quad y \quad y' = \sqrt{x^2 + 2y - 1} - 1 \quad y_p \quad y_{p'} \quad \frac{1}{2} h(y' + y_{p'})
\]

\[
\begin{array}{c|c|c|c|c|c|c}
1 & 1 & 0.414 & 1.0414 & 0.5142 & 0.0464 \\
1.1 & 1.0464 \\
\end{array}
\]

The difference in the (rounded) second decimal place between the values of \( x (1.1) \) in the two calculations suggests that the second decimal place cannot be relied upon in the first calculation.
B9. Trapezium rule calculation is:

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>m</th>
<th>mf_1(x)</th>
<th>mf_2(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2690</td>
<td>1</td>
<td>1.2690</td>
<td>1.2690</td>
</tr>
<tr>
<td>1.25</td>
<td>1.1803</td>
<td>2</td>
<td>2.3606</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0.9867</td>
<td>2</td>
<td>1.9734</td>
<td>1.9734</td>
</tr>
<tr>
<td>1.75</td>
<td>0.6839</td>
<td>2</td>
<td>1.3678</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2749</td>
<td>1</td>
<td>0.2749</td>
<td>0.2749</td>
</tr>
</tbody>
</table>

\[ I_1 = 3.5173 \times 0.5/2 = 0.8793 \text{ and } I_2 = 7.2457 \times 0.25/2 = 0.9057. \]

Difference table is:

\[
\begin{array}{cc}
-887 & -1049 \\
-1936 & -1092 \\
-3028 & -1062 \\
-4090 & \\
\end{array}
\]

\[ | \text{max truncation error} | = 1 \times 0.1092/12 = 0.009 \]

Hence \( I_2 = 0.91 \text{ or } 0.9. \)

Expect to reduce error by factor 4.

With \( n \) strips and step size \( 2h \), Taylor series for expansion of an integral \( I \) approximated by the trapezium rule is:

\[ I = I_n + C(2h)^2 + D(2h)^4 + \ldots = I_n + 4Ch^2 + 16Dh^4 + \ldots \quad \text{(a)} \]

With \( 2n \) strips and step size \( h \), we have:

\[ I = I_{2n} + Ch^2 + Dh^4 + \ldots \quad \text{(b)} \]

4(b) - (a) gives \( 3I = 4I_{2n} - I_n - 12Dh^4 + \ldots \)

i.e. \( I = (4I_{2n} - I_n)/3 = I_{2n} + (I_{2n} - I_n)/3 \)

\[ I_R = (4 \times 0.9057 - 0.8793)/3 = 0.914 \]

page 5
B10. Gradient of \( y = f(x) \) at \( x_0 \) is \( f''(x_0) = \frac{f(x_0)}{x_0 - x_1} \).

Hence \( x_1 - x_0 = \frac{f(x_0)}{f''(x_0)} \), i.e. \( x_1 = x_0 - \frac{f(x_0)}{f''(x_0)} \).

Likewise \( x_2 = x_1 - \frac{f(x_1)}{f''(x_1)} \) and in general \( x_{n+1} = x_n - \frac{f(x_n)}{f''(x_n)} \).

\[ f(x) = e^{-x} + x^4 - 2x^3 - 5x^2 - 1 \] and \( f''(x) = -e^{-x} + 4x^3 - 6x^2 - 10x; \) \( x_0 = 3.5 \)

Root is 3.47 (2 decimal places).

In a situation such as diagrammed, the Newton-Raphson method depends for convergence on the point of intersection of tangent with x-axis being closer to the root than the initial point. In the interval \([-0.3, 0]\) there must be a TV of \( f(x) \) so that \( f''(x) = 0 \) and the point of intersection may be far from initial point; so iteration may lead to a different root.

For bisection, \( f(-1.1) = 0.080 \);

\[ f(-1) = -0.281 \]
\[ f(-1.05) = -0.124 \]
\[ f(-1.075) = -0.208; \]

\[ f(-1.0875) = 0.024 \]

Hence root lies in \([-1.0875, -1.075]\).
2004 Applied Mathematics

Advanced Higher – Section C

Finalised Marking Instructions
Advanced Higher Applied Mathematics 2004
Solutions for Section C (Mechanics 1 and 2)

C1. \[ r(t) = (2t^2 - t)i - (3t + 1)j \]
\[ \Rightarrow v(t) = (4t - 1)i - 3j \]
\[ \Rightarrow |v(t)| = \sqrt{(4t - 1)^2 + 9} \]

When the speed is 5,
\[ (4t - 1)^2 + 9 = 25 \]
\[ (4t - 1)^2 = 16 \]
\[ 4t - 1 = \pm 4 \]
\[ t = \frac{5}{4} \text{ seconds (as } t > 0) \]

C2. (a) \[ v_F = 25\sqrt{2}(\cos 45^\circ i + \sin 45^\circ j) \]
\[ = 25(i + j) \]
\[ r_F = 25t(i + j) \text{ as } r_F(0) = 0 \]
\[ v_L = 20j \]
\[ r_L = 20tj + c \]

But \( r_L(0) = 10i \) so \( r_L = 10i + 20tj \)

The position of the ferry relative to the freighter is
\[ r_F - r_L = (25t - 10)i + 5tj \]

(b) When \( t = 1 \)
\[ |r_F - r_L| = \sqrt{15^2 + 5^2} \]
\[ = \sqrt{250} = 5\sqrt{10} \text{ km} \]

C3. (a) Using \( T = \frac{2\pi}{\omega} \Rightarrow 8\pi = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{1}{4} \).

Maximum acceleration = \( \omega^2 a \)
\[ \frac{1}{4} = \frac{1}{16} a \Rightarrow a = 4 \]

(b) Maximum speed = \( \omega a = \frac{1}{4} \times 4 = 1 \).

Using
\[ v^2 = \omega^2(a^2 - x^2) \]
\[ \left(\frac{1}{2}\right)^2 = \frac{1}{16}(16 - x^2) \]
\[ 4 = 16 - x^2 \]
\[ x^2 = 12 \]
\[ x = \pm 2\sqrt{3} \text{ m} \]

---

page 2
C4.

Resolving perp. to plane: \( R = mg \cos \theta \)
Parallel to the plane (by Newton II)

\[
m a = -\mu R - mg \sin \theta \\
= -\mu mg \cos \theta - mg \sin \theta \\
a = -g (\mu \cos \theta + \sin \theta) \\
= \frac{-(5 + 12\mu)g}{13}
\]

Using \( v^2 = u^2 + 2as \)

\[
0 = gL - \frac{2(5 + 12\mu)gL}{13}
\]

\[
gL = \frac{2(5 + 12\mu)gL}{13}
\]

\[
10 + 24\mu = 13 \Rightarrow \mu = \frac{1}{8}
\]

C5.

Combined mass = \( M + 0.01M = 1.01M \).
By Newton II

\[
1.01Ma = (P + 0.05P) - 1.01Mg
\]

\[
1.01Ma = 1.05Mg - 1.01Mg
\]

\[
a = 0.04g
\]

\[
a = \frac{4}{101^2} g (\approx 0.3) \text{ m s}^{-2}
\]
C6. (i) By conservation of energy, the speed of block A \(v_A\) immediately before the collision is given by
\[
v_A = \sqrt{2gh}.
\]
By conservation of momentum, the speed of the composite block \(v_C\) after the collision is given by
\[
2mv_C = mv_A
\]
\[
v_C = \frac{1}{2}\sqrt{2gh}
\]
(ii) By the work/energy principle

\[
F \times h = \frac{1}{2}(2m).\frac{1}{8}2gh + 2mg \times \frac{1}{4}h
\]
\[
F = \frac{mg}{2} + mg
\]
\[
F = \frac{3}{2}W \text{ since } W = mg.
\]

C7. (a) The equations of motion give
\[
\dot{y} = -g \\
\dot{v}(0) = V \cos \alpha \hat{i} + V \sin \alpha \hat{j}
\]
\[
\ddot{y} = -gt + V \sin \alpha
\]
\[
y = V \sin \alpha t - \frac{1}{2}gt^2
\]

Maximum height when \(\dot{y} = 0 \Rightarrow t = \frac{V}{g} \sin \alpha\), and so
\[
H = V \sin \alpha \times \frac{V}{g} \sin \alpha - \frac{1}{2}g \frac{V^2}{g^2} \sin^2 \alpha
\]
\[
= \frac{V^2}{2g} \sin^2 \alpha
\]
(b) (i)
\[
h = \frac{V^2}{2g} \sin^2 2\alpha
\]
\[
= \frac{V^2}{2g} 4 \sin^2 \alpha \cos^2 \alpha
\]
\[
= \frac{2V^2}{g} \sin^2 \alpha (1 - \sin^2 \alpha)
\]
\[
= 4H \left(1 - \frac{2gH}{V^2}\right) \text{ since } \sin^2 \alpha = \frac{2gH}{V^2}
\]
(ii) Since \(h = 3H\)
\[
3H = 4H \left(1 - \sin^2 \alpha\right)
\]
\[
\frac{3}{4} = 1 - \sin^2 \alpha
\]
\[
\sin^2 \alpha = \frac{1}{4}
\]
\[
\sin \alpha = \pm \frac{1}{2}
\]
\[
\Rightarrow \alpha = \frac{\pi}{6} \text{ and so } 2\alpha = \frac{\pi}{3}
\]
C8. (a) Radius of horizontal circle \( r = L \sin 60^\circ = \frac{\sqrt{3}}{2} L \).

\[
AB = \frac{r}{\sin 30^\circ} = 2 \times \frac{\sqrt{3}}{2} L = \sqrt{3} L
\]

Extension of \( AB, x = (\sqrt{3} - 1)L \)

Tension in \( AB, T_1 = \frac{Lx}{L} = 2(\sqrt{3} - 1)mg. \)

(b) Resolving vertically (where \( T_2 \) is the tension in \( BC \))

\[
T_1 \cos 30^\circ = mg + T_2 \cos 60^\circ
\]

\[
\frac{\sqrt{3}}{2} \times 2(\sqrt{3} - 1)mg = mg + \frac{1}{2}T_2
\]

\[
T_2 = (6 - 2\sqrt{3} - 2)mg
\]

\[
= 2(2 - \sqrt{3})mg
\]

(c) Resolving horizontally (using \( L = 1 \))

\[
T_1 \sin 30^\circ + T_2 \sin 60^\circ = m\left(\frac{\sqrt{3}}{2}\right)\omega^2
\]

\[
\frac{1}{2} \times 2(\sqrt{3} - 1)mg + \frac{\sqrt{3}}{2} \times 2(2 - \sqrt{3})mg = m\left(\frac{\sqrt{3}}{2}\right)\omega^2
\]

\[
(2\sqrt{3} - 2 + 4\sqrt{3} - 6)g = \sqrt{3}\omega^2
\]

\[
(6\sqrt{3} - 8)g = \sqrt{3}\omega^2
\]

\[
\omega^2 = \frac{2(3\sqrt{3} - 4)g}{\sqrt{3}}
\]

\[
\omega = \sqrt{\frac{2(3\sqrt{3} - 4)g}{\sqrt{3}}}
\]
C9. (i)

\[ \frac{dv}{dt} = -mkv^3 \]
\[ v \frac{dv}{dx} = -kv^3 \]
\[ \frac{dv}{dx} = -kv^2 \]

Separating the variables and integrating gives
\[ \int v^{-2} dv = \int -k dx \]
\[ \Rightarrow -v^{-1} = -kx + c \]

At \( x = 0, v = U \)
\[ -U^{-1} = c \]
so
\[ v^{-1} = kx + U^{-1} \]
\[ v = \frac{U}{1 + kUx} \]

(ii) Now \( v = \frac{dx}{dt} \), so
\[ \frac{dx}{dt} = \frac{U}{1 + kUx} \]
\[ \int (1 + kUx) dx = \int U dt \]
\[ x + \frac{1}{2}kUx^2 = Ut + c_1 \]
Since \( x = 0 \) when \( t = 0 \), then \( c_1 = 0 \)
\[ kUx^2 + 2x = 2Ut \]

(iii)
\[ V = \frac{1}{2}U \Rightarrow \frac{1}{2}U (1 + kUx) = U \]
\[ \Rightarrow 1 + kUx = 2 \Rightarrow x = \frac{1}{kU} \]

The time taken
\[ 2Ut = kU \frac{1}{k^2U^2} + \frac{2}{kU} = \frac{3}{kU} \]
\[ \Rightarrow t = \frac{3}{2kU^2} \]
2004 Applied Mathematics

Advanced Higher – Section D

Finalised Marking Instructions
Advanced Higher Applied Mathematics 2004
Solutions for Section D (Mathematics 1)

D1. \[(4x - 5y)^4 = (4x)^4 - 4 \times (4x)^3(5y) + 6 \times (4x)^2(5y)^2 - 4 \times (4x)(5y)^3 + (5y)^4 \]
\[= 256x^4 - 1280x^3y + 2400x^2y^2 - 2000xy^3 + 625y^4,\]

When \(y = \frac{1}{x}\), the term independent of \(x\) is 2400.

D2. \[y = x^2 \ln x\]
\[\frac{dy}{dx} = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x\]

Thus \(k = 2\).

D3. \[(a)\]

\[
\begin{array}{ccc|c}
1 & 1 & -2 & -6 \\
3 & -1 & 1 & 7 \\
2 & 1 & -\lambda & -2 \\
\hline
1 & 1 & -2 & -6 \\
0 & -4 & 7 & 25 \\
0 & -1 & 4 - \lambda & 10 \\
\hline
1 & 1 & -2 & -6 \\
0 & -4 & 7 & 25 \\
0 & 0 & 9 - 4\lambda & 15 \\
\end{array}
\]

There is no solution when \(\lambda = \frac{9}{4}\).

(b) When \(\lambda = 1\),

5c = 15 \(\Rightarrow\) c = 3

\(-4b + 21 = 25 \Rightarrow b = -1\)

\(a - 1 - 6 = -6 \Rightarrow a = 1\)

i.e. \(a = 1, b = -1, c = 3\)

2E1
D4. \[ x + 1 = u \Rightarrow dx = du \]
\[ x = u - 1 \Rightarrow x^2 + 2 = u^2 - 2u + 3. \]
\[
\int \frac{x^2 + 2}{(x + 1)^2} \, dx = \int \frac{u^2 - 2u + 3}{u^2} \, du
\]
\[
= \int 1 - \frac{2}{u} + 3u^{-2} \, du
\]
\[
= u - 2 \ln |u| - 3u^{-1} + c
\]
\[
= x - 2 \ln |x + 1| - \frac{3}{x + 1} + c
\]

1 for differentials
1 for substitution
1 for simplifying

D5. (a) \[
\frac{(x - 1)(x - 4)}{x^2 + 4} = A + \frac{Bx + C}{x^2 + 4}
\]
\[
x^2 - 5x + 4 = Ax^2 + 4A + Bx + C
\]
\[
A = 1, \quad B = -5, \quad C = 0
\]
\[\text{M1} \quad 2E1\]
\[
i.e. \quad f(x) = 1 - \frac{5x}{x^2 + 4}.
\]

(b) As \( x \to \pm \infty \), \( y \to 1 \).
[No vertical asymptotes since \( x^2 + 4 \neq 0 \).] 1

(c) \[
f(x) = 1 - \frac{5x}{x^2 + 4}
\]
\[
f''(x) = -\frac{5(x^2 + 4) - 10x^2}{(x^2 + 4)^2} = 0 \text{ at S.V.}
\]
\[
\Rightarrow 20 - 5x^2 = 0 \Rightarrow x = \pm 2
\]
\[
\Rightarrow (2, -\frac{1}{2}) \text{ and } (2, 2\frac{1}{2})
\]

(d) \( y = 0 \) \( \Rightarrow \) \( x = 1 \) or \( x = 4 \).
Area = \[-\int_1^4 \left( 1 - \frac{5x}{x^2 + 4} \right) \, dx \]
\[
= -\left[ x - \frac{5}{2} \ln(x^2 + 4) \right]_1^4
\]
\[
= -\left[ \frac{5}{2} \ln 4 - 3 \right] - \left[ 1 - \frac{5}{2} \ln 5 \right]
\]
\[
= \frac{5}{2} \ln 4 - 3 = 5 \ln 2 - 3 \text{ (acceptable but not required)}
\]
\[
= 0.47 \text{ (acceptable but not required)}
\]
2004 Applied Mathematics

Advanced Higher – Section E

Finalised Marking Instructions
E1.  
(a) Stratified
and Quota [or Quota (convenience)]
(b) Approach (a) should be best
since (b) is not random (other forms e.g. Glasgow not typical, biased)

E2.  
(a) $F \sim Bin(192, 0.002)$.  
1 for distribution
1 for parameters

(b) 
$$P(F \geq 3) = 1 - P(F \leq 2)$$
$$= 1 - (0.6809 + 0.2620 + 0.0501)$$
$$= 0.0070$$

Notes: applying a Poisson distribution loses (at least) one mark; a Normal
distribution loses two marks.

(c) Approximate using the $Poi(0.384)$
1 for distribution
1 for parameters

E3.  
Assume that yields are normally distributed.
Assume that the standard deviation is unchanged.
$$\bar{x} = 404.2,$$
A 95% confidence interval for the mean yield, $\mu$, is given by:
$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$$
$$404.2 \pm 1.96 \frac{10}{\sqrt{5}}$$
$$404.2 \pm 8.8$$
or $(395.4, 413.0)$.
The fact that the confidence interval does not include 382
provides evidence, at the 5% level of significance, of a
change in the mean yield.

E4.  
TNE = 3% of 500 = 15
With maximum allowable standard deviation
$$P(\text{weight} < 485) = 0.025$$
$$\Rightarrow \frac{485 - 505}{\sigma} = -1.96$$
$$\Rightarrow \sigma = \frac{20}{1.96} = 10.2$$

There will be a small probability of obtaining a content
weight less than 470g with the normal model.

page 2
(a) \[ P(\text{Alaskan fish classified as Canadian}) \]
\[ = P(X > 120 \mid X \sim N(100,20^2)) \]
\[ = P\left(Z > \frac{120 - 100}{20}\right) \]
\[ = P(Z > 1) \]
\[ = 0.1587 \]

(b) The probability is the same as in (a) because of symmetry.

(c) \[ P(\text{Canadian origin} \mid \text{Alaskan predicted}) \]
\[ = \frac{P(\text{Alaskan predicted and Canadian origin})}{P(\text{Alaskan predicted})} \]
\[ = \frac{P(\text{Alaskan predicted but Canadian origin})}{P(\text{Ala pred and Alaskan}) + P(\text{Ala pred but Canadian})} \]
\[ = \frac{0.6 \times 0.8413 + 0.4 \times 0.1587}{0.06348} \]
\[ = \frac{0.50478 + 0.06348}{0.112} \]
\[ = 0.112. \]

*Note: Alternative methods acceptable e.g. Venn or Tree Diagrams*
2004 Applied Mathematics

Advanced Higher – Section F

Finalised Marking Instructions
Advanced Higher Applied Mathematics 2004
Solutions for Section F (Numerical Analysis 1)

F1. \[ f(x) = \ln(2 - x) \quad f'(x) = \frac{-1}{2 - x} \quad f''(x) = \frac{-1}{(2 - x)^2} \quad f'''(x) = \frac{-2}{(2 - x)^3} \]

Taylor polynomial is

\[ p(1 + h) = \ln 1 - h - \frac{h^2}{2} - \frac{2h^3}{6} \]

\[ = -h - \frac{h^2}{2} - \frac{h^3}{3} \]

For \( \ln 1.1, h = -0.1 \) and \( p(0.9) = 0.1 - 0.005 + 0.00033 = 0.0953. \)

\[ p(a + h) = \ln(2 - a) - \frac{1}{2 - a} h \]

Hence expect \( f(x) \) to be more sensitive in \( L_2 \) since coefficient of \( h \) is much larger.

F2. \[ L(2.5) \]

\[ = \frac{(2.5 - 1.5)(2.5 - 3.0)(2.5 - 4.5)}{(0.5 - 0.5)(0.5 - 3.0)(0.5 - 4.5)} \cdot 1.737 + \frac{(2.5 - 0.5)(2.5 - 3.0)(2.5 - 4.5)}{(1.5 - 0.5)(1.5 - 3.0)(1.5 - 4.5)} \cdot 2.412 \]

\[ + \frac{(2.5 - 0.5)(2.5 - 1.5)(2.5 - 4.5)}{(3.0 - 0.5)(3.0 - 1.5)(3.0 - 4.5)} \cdot 3.284 + \frac{(2.5 - 0.5)(2.5 - 1.5)(2.5 - 3.0)}{(4.5 - 0.5)(4.5 - 1.5)(4.5 - 3.0)} \cdot 2.797 \]

\[ = \frac{1 \times 1.737}{10} + \frac{2 \times 2.412}{4.5} + \frac{4 \times 3.284}{2.5 \times 2.25} - \frac{1 \times 2.797}{18} \]

\[ = -0.1737 + 1.0720 + 2.3353 - 0.1554 = 3.078 \]

F3. \[ \Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0 \]

Maximum rounding error \( = \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon. \)

\[ \Delta^2 f_0 = 2.618 - 2 \times 2.369 + 2.124 = 0.004 \]

and \( 4\varepsilon = 4 \times 0.0005 = 0.002. \)

\( \Delta^2 f_0 \) appears to be significantly different from 0.
\textbf{F4.}  
(a) Difference table is:

$$
\begin{array}{|c|c|c|c|c|}
\hline
i & x & f(x) & \text{diff1} & \text{diff2} \\
\hline
0 & 0 & 1.023 & 352 & -95 \\
1 & 0.5 & 1.375 & 257 & -92 \\
2 & 1 & 1.632 & 165 & -96 \\
3 & 1.5 & 1.797 & 69 & \\
4 & 2 & 1.866 & & \\
\hline
\end{array}
$$

(b) \( p = 0.3 \)

\[
f(0.65) = 1.375 + 0.3(0.257) + \frac{(0.3)(-0.7)}{2} (-0.092)
\]

\[
= 1.375 + 0.077 + 0.010 = 1.462
\]

(or, with \( p = 1.3, 1.023 + 0.458 - 0.019 \)).

\textbf{F5.}  
Trapezium rule calculation is:

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & f(x) & m & mf_1(x) & mf_2(x) \\
\hline
1 & 1.2690 & 1 & 1.2690 & 1.2690 \\
1.25 & 1.1803 & 2 & 2.3606 & \\
1.5 & 0.9867 & 2 & 1.9734 & 1.9734 \\
1.75 & 0.6839 & 2 & 1.3678 & \\
2 & 0.2749 & 1 & 0.2749 & 0.2749 \\
\hline
\end{array}
\]

Hence \( I_1 = 3.5173 \times 0.5/2 = 0.8793 \) and \( I_2 = 7.2457 \times 0.25/2 = 0.9057. \)

Difference table is:

\[
\begin{array}{|c|c|}
\hline
\text{max truncation error} & 1 \\
\hline
-1.215 & \frac{1}{12} \times 0.009 \\
-1.192 & 0.91 \\
-3128 & 0.91 \\
-4090 & \\
\hline
\end{array}
\]

Expect to reduce error by factor 4.

With \( n \) strips and step size \( 2h \), Taylor series for expansion of an integral \( I \)
approximated by the trapezium rule is:

\[ I = I_n + C(2h)^2 + D(2h)^4 + \ldots = I_n + 4Ch^2 + 16Dh^4 + \ldots \]  

(a)

With \( 2n \) strips and step size \( h \), we have: \( I = I_{2n} + Ch^2 + Dh^4 + \ldots \)  

(b)

4(b) - (a) gives \( 3I = 4I_{2n} - I_n - 12Dh^4 + \ldots \)

i.e. \( I = (4I_{2n} - I_n)/3 = I_{2n} + (I_{2n} - I_n)/3 \)

\( \therefore I_R = (4 \times 0.9057 - 0.8793)/3 = 0.914 \)
2004 Applied Mathematics

Advanced Higher – Section G

Finalised Marking Instructions
G1. \[
\mathbf{r}(t) = (2t^2 - t)i - (3t + 1)j
\]
\[\Rightarrow \mathbf{v}(t) = (4t - 1)i - 3j\]
\[\Rightarrow |\mathbf{v}(t)| = \sqrt{(4t - 1)^2 + 9}\]

When the speed is 5,
\[(4t - 1)^2 + 9 = 25\]
\[(4t - 1)^2 = 16\]
\[4t - 1 = \pm 4\]
\[t = \frac{5}{4} \text{ seconds (as } t > 0).\]

G2. (a) \[
\mathbf{v}_F = 25\sqrt{2}(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j})
\]
\[= 25(i + j)\]
\[\mathbf{r}_F = 25t(i + j) \quad \text{as } \mathbf{r}_F(0) = 0\]
\[\mathbf{v}_L = 20j\]
\[\mathbf{r}_L = 20tj + c\]

But \(\mathbf{r}_L(0) = 10\mathbf{i}\) so \(\mathbf{r}_L = 10\mathbf{i} + 20tj\)

The position of the ferry relative to the freighter is
\[
\mathbf{r}_F - \mathbf{r}_L = (25t - 10)i + 5tj
\]

(b) When \(t = 1\)
\[
|\mathbf{r}_F - \mathbf{r}_L| = \sqrt{15^2 + 5^2}
\]
\[= \sqrt{250} = 5\sqrt{10} \text{ km}\]

G3.

Combined mass = \(M + 0.01M = 1.01M\).

By Newton II
\[
1.01Ma = (P + 0.05P) - 1.01Mg
\]
\[1.01Ma = 1.05Mg - 1.01Mg\]
\[1.01a = 0.04g\]
\[a = \frac{4}{101}g (= 0.3) \text{ m s}^{-2}\]
Resolving perp. to plane: \( R = mg \cos \theta \)
Parallel to the plane (by Newton II)

\[
ma = -\mu R - mg \sin \theta \\
= -\mu mg \cos \theta - mg \sin \theta
\]

\[
a = -g (\mu \cos \theta + \sin \theta) \\
= \frac{-(5 + 12\mu)g}{13}
\]

Using \( v^2 = u^2 + 2as \)

\[
0 = gL - \frac{2(5 + 12\mu)gL}{13}
\]

\[
gL = \frac{2(5 + 12\mu)gL}{13}
\]

\[
10 + 24\mu = 13 \Rightarrow \mu = \frac{1}{8}
\]
G5. (a) The equations of motion give
\[ \ddot{y} = -g \quad \quad \dot{v}(0) = V \cos \alpha i + V \sin \alpha j \]
\[ \ddot{y} = -gt + V \sin \alpha \]
\[ y = V \sin \alpha t - \frac{1}{2}gt^2 \]

Maximum height when \( \ddot{y} = 0 \) \( \Rightarrow \quad t = \frac{V}{g} \sin \alpha \), and so
\[ H = V \sin \alpha \times \frac{V}{g} \sin \alpha - \frac{1}{2}g \frac{V^2}{g^2} \sin^2 \alpha \]
\[ = \frac{V^2}{2g} \sin^2 \alpha \]

(b) (i)
\[ h = \frac{V^2}{2g} \sin^2 2\alpha \]
\[ = \frac{V^2}{2g} \frac{4}{4} \sin^2 \alpha \cos^2 \alpha \quad 1 \]
\[ = \frac{2V^2}{g} \sin^2 \alpha (1 - \sin^2 \alpha) \]
\[ = 4H \left(1 - \frac{2gH}{V^2}\right) \quad \text{since } \sin^2 \alpha = \frac{2gH}{V^2} \]

(ii) Since \( h = 3H \)
\[ 3H = 4H \left(1 - \sin^2 \alpha\right) \]
\[ \frac{3}{4} = 1 - \sin^2 \alpha \]
\[ \sin^2 \alpha = \frac{1}{4} \]
\[ \sin \alpha = \pm \frac{1}{2} \]
\[ \Rightarrow \alpha = \frac{\pi}{6} \quad \text{and so } 2\alpha = \frac{\pi}{3} \]