Section F (Numerical Analysis 1)

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

F1. The function \( f \) is defined for \( x > 0 \cdot 8 \) by \( f(x) = \frac{1}{x - 4} \).

The polynomial \( p \) is the Taylor polynomial of degree two for the function \( f \) near \( x = 1 \). Express \( p(1 + h) \) in the form \( c_0 + c_1h + c_2h^2 \).

Use this polynomial to estimate the value of \( f(0.99) \).

State, with a reason, whether or not \( f(x) \) is sensitive to small changes in \( x \) in the neighbourhood of \( x = 1 \).

F2. The following data are available for a function \( f \):

\[
\begin{array}{c|c|c|c}
  x & 0 & 2 & 5 \\
  f(x) & 1.3271 & 1.5238 & 1.8516 \\
\end{array}
\]

Use the quadratic Lagrange interpolation formula to estimate \( f(3) \).

F3. In the usual notation for forward differences of function values \( f(x) \) tabulated at equally spaced values of \( x \),

\[ \Delta_i f_i = f_{i+1} - f_i, \]

where \( f_i = f(x_i) \) and \( i = -2, -1, 0, 1, 2, \ldots \)

Show that \( \Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0 \).

If each value of \( f_i \) is subject to an error whose magnitude is less than or equal to \( \varepsilon \), determine the magnitude of the maximum possible rounding error in \( \Delta^3 f_0 \).

Rounded values of a function \( f \) are known to be \( f_0 = 1.311, f_1 = 1.416, f_2 = 1.532, f_3 = 1.658 \). Obtain \( \Delta^3 f_0 \) and the magnitude of the maximum rounding error in \( \Delta^3 f_0 \).

Hence state whether or not this third difference appears to be significantly different from zero.

F4. The following data (accurate to the degree implied) are available for a function \( f \):

\[
\begin{array}{c|c|c|c|c|c|c}
  x & 1.0 & 1.1 & 1.2 & 1.3 & 1.4 & 1.5 \\
  f(x) & 1.263 & 1.456 & 1.696 & 1.991 & 2.351 & 2.782 \\
\end{array}
\]

(a) Construct a difference table of fourth order for the data.

(b) Using the Newton forward difference formula of degree three, and working to three decimal places, obtain an approximation to \( f(1.18) \).
F5. (a) Using a Taylor polynomial of degree two, or otherwise, derive the trapezium rule over a single strip and the corresponding principal error term.

(b) Use the composite trapezium rule with four strips to obtain an estimate for the integral

\[ \int_{1}^{4} x^3 \ln x \, dx. \]

Perform the calculations using four decimal places.

(c) Given that for \( f(x) = x^3 \ln x \), \( f''(x) = 2 \ln x + 3 \), obtain the maximum value of \( x^3 \ln x \) on the interval \([1, 4]\) and hence obtain an estimate of the maximum truncation error in the integral.

Hence state the value of the integral to a suitable accuracy.