# X106/701

NATIONAL QUALIFICATIONS 2004

MONDAY, 24 MAY
1.00 PM – 4.00 PM

# APPLIED MATHEMATICS
## ADVANCED HIGHER

**Read carefully**

1. Calculators may be used in this paper.
2. There are seven Sections in this paper.

   Section A assesses the units Statistics 1 and 2
   Section B assesses the units Numerical Analysis 1 and 2
   Section C assesses the units Mechanics 1 and 2
   Section D assesses the unit Mathematics 1
   Section E assesses the unit Statistics 1
   Section F assesses the unit Numerical Analysis 1
   Section G assesses the unit Mechanics 1.

   Candidates must attempt Section A (Statistics 1 and 2) **or** Section B (Numerical Analysis 1 and 2) **or** Section C (Mechanics 1 and 2)

   **and one** of the following Sections:

   Section D (Mathematics 1)
   Section E (Statistics 1)
   Section F (Numerical Analysis 1)
   Section G (Mechanics 1).
3. **Candidates must use a separate answer book for each Section.** Take care to show clearly the section chosen. On the front of the answer book, in the top right hand corner, write A, B, C, D, E, F or G.
4. A booklet of Mathematical Formulae and Statistical Tables is supplied for all candidates. It contains Numerical Analysis formulae and Statistical formulae and tables.
5. Candidates are not allowed to attempt questions in the option in the same subject area as their chosen core, eg candidates doing Section A (Statistics 1 and 2) may not choose Section E (Statistics 1).
6. **Full credit will be given only where the solution contains appropriate working.**

SCOTTISH QUALIFICATIONS AUTHORITY

*Marks*

## Section A (Statistics 1 and 2)

**ONLY candidates doing the course Statistics 1 and 2 and one unit chosen from Mathematics 1 (Section D), Numerical Analysis 1 (Section F) and Mechanics 1 (Section G) should attempt this Section.**

**Answer all the questions.**

**Answer these questions in a separate answer book, showing clearly the section chosen.**

**A1.** A discount warehouse company has branches throughout Scotland. Customers (trade and private) must be account holders and there are four times as many trade customers as private customers. In order to obtain data from customers on a variety of issues, senior management discussed the following approaches.

(*a*) Carry out telephone interviews of 80 trade and 20 private customers selected at random from the database of all customers held at head office.

(*b*) Interview 80 trade and 20 private customers leaving the Glasgow branch.

State the type of sampling involved in each approach.

State which approach is likely to yield the better estimates of parameters of the population of customers and why you rejected the other approach. **4**

**A2.** A budget airline uses Boeing 737 aircraft with 189 seats. In management discussion of seat reservation strategies, it was assumed that each passenger with a reserved seat has probability 0·002 of failing to show up for the flight. Suppose that a strategy of selling 192 seats for such flights was to be adopted.

(*a*) State the distribution of the number of passengers who fail to show up for a fully booked flight. **2**

(*b*) Obtain the probability that a fully booked flight can depart without any passengers with reserved seats having to be left behind. **3**

(*c*) Specify the distribution which would be appropriate to use if only an estimate of this probability is required. **2**

**A3.** Prior to a major maintenance of a batch chemical production plant, the mean yield was 382 kg per batch. When production was resumed, a random sample of yields (kg) was:

418 411 401 394 397.

Stating any assumption required, obtain a 95% confidence interval for the mean yield following maintenance.

Comment on the effect of the maintenance on the mean yield. **6**

*Marks*

**A4.** Under the Weights and Measures (Packaged Goods) Regulations 1986, display of the text **500 g℮** on a packet of cereal requires the packer to ensure that:

(i) the actual contents of the packets shall not be less, on average, than 500 g;

(ii) not more than 2·5% of packets may be non-standard ie contain less than the stated content weight by more than the Tolerable Negative Error (TNE) specified.

Given that the TNE for a packet displaying **500 g℮** is 3% and that content weight is normally distributed, calculate the maximum allowable standard deviation when the mean content weight is set at 505 g. **5**

A third legal requirement is that no package may be sub-standard ie deficient by more than twice the TNE. Explain why this requirement cannot be satisfied in the above scenario. **1**

**A5.** A manufacturer of Automatic Teller Machines introduced changes to the procedure for installing the printer in the carcass as part of a process improvement initiative.

Random samples of installation times for a technician before and after the changes are tabulated and partially ranked below (where A = After and B = Before).

| *Time* | 19 | 29 | 31 | 35 | 37 | 39 | 39 | 41 | 42 | 43 | 45 | 52 | 59 | 64 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Period* | A | A | B | A | A | B | A | A | B | B | A | B | B | B |
| *Rank* | 1 | 2 | 3 | 4 | 5 | | | | | | | | | |

Stating any assumption required, determine whether or not the data provide evidence that median installation time has been reduced. **6**

**A6.** A cosmetics manufacturer undertook a market research survey to investigate customer preference for three brands of its moisturising creams. The survey involved taking a random sample of 240 purchasers of the creams and classifying them by brand as follows.

| *Cream purchased* | A | B | C |
|---|---|---|---|
| *Number of purchasers* | 66 | 99 | 75 |

Carry out a chi-squared test, at the 5% significance level, of the null hypothesis that customers have a uniform distribution of preferences for the creams. **5**

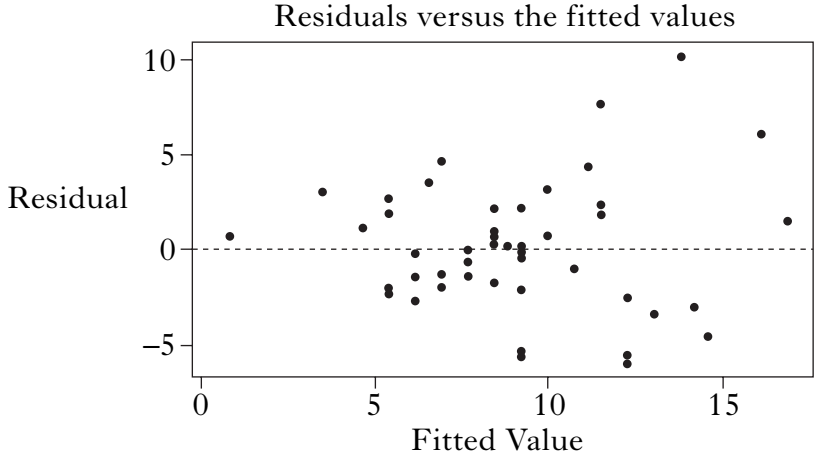
A spreadsheet package gives the p-value for the test to be 0·026. Explain how this value supports your earlier conclusion. **1**

*Marks*

**A7.** As part of a research project into the energy budgets of grey squirrels, a biologist weighed and measured the heights of a random sample of 50 chestnuts. The aim was to enable prediction of the food intake of squirrels from measurements made of partially eaten chestnuts. A scatter plot of weight ($W$ grammes) against height ($h$ cm) suggested that a linear model was appropriate. The equation of the least squares regression line was $W = -6{\cdot}81 + 7{\cdot}63h$.

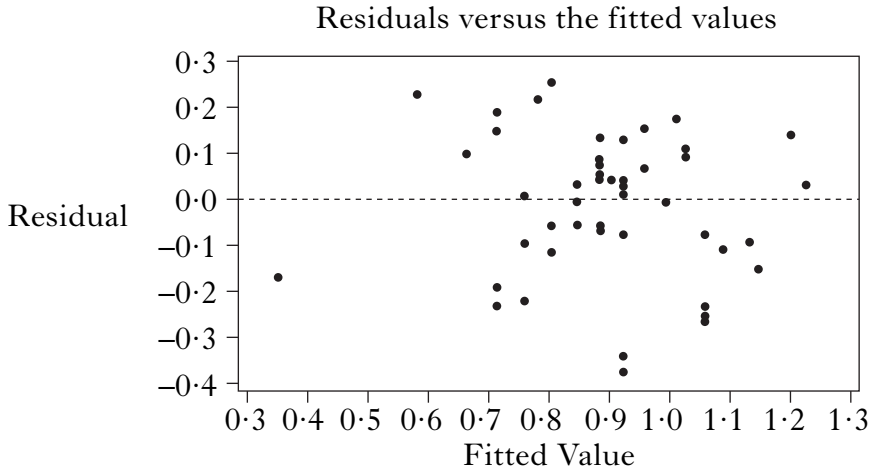
(*a*) For a chestnut with $h = 2{\cdot}7$ and $W = 24{\cdot}0$, calculate the fitted value and residual for her model. **2**

(*b*) Of the assumptions required in order to use a linear model, state which, in view of the residual plot below, is violated. **1**




The weights and the heights were transformed by taking logarithms giving the variables $Y$ and $x$ respectively.

(*c*) Comment on the residual plot below which resulted from fitting the linear model $Y_i = \alpha + \beta x_i + \varepsilon_i$. **1**




(*d*) State the assumption required, in addition to the one you have referred to in (*b*), for use of the second model to calculate prediction intervals. **1**

*Marks*

**A8.** The PCS-12 is a generic measure of physical health status. The table below gives PCS-12 scores for a random sample of patients immediately prior to having a hip replacement (Pre) and six months later (Post).

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pre | 36 | 45 | 30 | 63 | 48 | 52 | 44 | 44 | 45 | 51 | 39 | 44 |
| Post | 39 | 42 | 33 | 70 | 53 | 51 | 48 | 51 | 51 | 51 | 42 | 50 |

(*a*) Perform a hypothesis test, stating any assumption required, to investigate whether or not the data provide evidence that the median difference (Post – Pre) is positive. **6**

(*b*) On the assumption that the population of post-operation PCS-12 scores is normally distributed with mean 50 and standard deviation 10, demonstrate that the above post-operation PCS-12 scores are consistent with the mean for the general population. **5**

**A9.** Marine biologists studied samples of tagged salmon of both Alaskan and Canadian origin. From measurements of the growth ring diameter, $X$, of scales for the period of time the fish lived in fresh water, they established that $X$ could be modelled by N (100, $20^2$) for fish of Alaskan origin and by N (140, $20^2$) for fish of Canadian origin.

In order to predict the origin of untagged fish caught, the biologists adopted the following rule:

**Predict fish to be of Alaskan origin if $X < 120$,**
**otherwise predict fish to be of Canadian origin.**

(*a*) Calculate the probability that application of this rule would lead to a salmon of Alaskan origin being misclassified as being of Canadian origin. **3**

(*b*) State, with justification, the probability that application of the rule would lead to a salmon of Canadian origin being misclassified as being of Alaskan origin. **1**

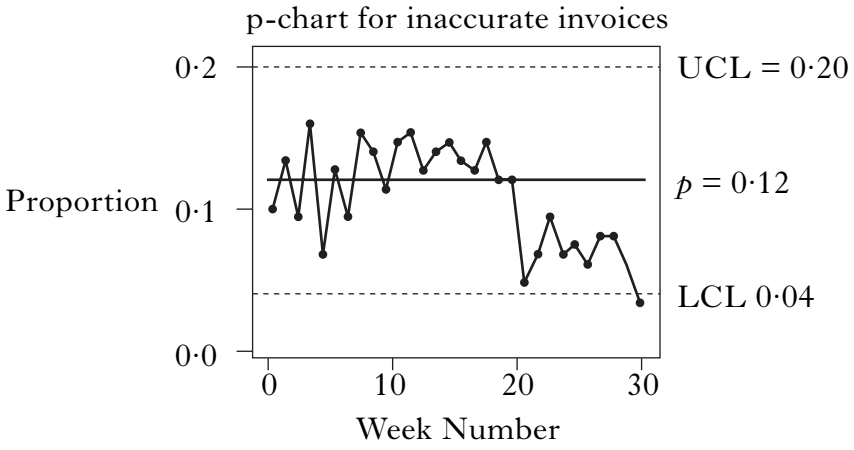
(*c*) Given that 60% of salmon in the area are of Alaskan origin, calculate the probability that a salmon, predicted by the rule to be of Alaskan origin, is actually of Canadian origin. **5**

**[Turn over**

*Marks*

**A10.** The proportion, $p$, of inaccurate invoices prepared by a large mail order company is a constant. If the random variable $X$ represents the number of inaccurate invoices in samples of size $n$, show that the proportion $\frac{X}{n}$ has standard deviation $\sqrt{\frac{p(1-p)}{n}}$. **3**

Following complaints from customers about inaccurate invoices, the department manager estimated the proportion, $p$, to be 0·12. He then initiated monitoring of the invoice preparation process using a Shewhart chart, with 3-sigma limits, for the proportion of inaccurate invoices in weekly random samples of 150 of the invoices prepared during that week. A copy of the chart after 30 weeks is shown below.




(*a*) Confirm the chart limits. **2**

(*b*) Explain how the chart provides evidence that steps taken to improve the accuracy of the invoicing process, introduced at the end of Week 20, have been effective. **1**

(*c*) What advice would you give on any future monitoring of the process using a p-chart? **2**

[*END OF SECTION A*]

**All candidates who have attempted Section A (Statistics 1 and 2) should now attempt ONE of the following**

**Section D (Mathematics 1) on *Page fifteen***

**Section F (Numerical Analysis 1) on *Pages eighteen* and *nineteen***

**Section G (Mechanics 1) on *Pages twenty* and *twenty-one.***

*Marks*

## Section B (Numerical Analysis 1 and 2)

**ONLY candidates doing the course Numerical Analysis 1 and 2 and one unit chosen from Mathematics 1 (Section D), Statistics 1 (Section E) and Mechanics 1 (Section G) should attempt this Section.**

**Answer all the questions.**

**Answer these questions in a separate answer book, showing clearly the section chosen.**

**B1.** The polynomial $p$ is the Taylor polynomial of degree three for a function $f$ near $x = 1$.

Given the function $f(x) = \ln(2 - x)$, where $x < 2$, express $p(1 + h)$ in the form $c_0 + c_1h + c_2h^2 + c_3h^3$. **2**

Estimate the value of $\ln(1{\cdot}1)$ using this approximation, giving your answer to four decimal places. **2**

Write down the first degree Taylor polynomial for $f$ near $x = a$ where $a < 2$. Given the intervals $I_1[0{\cdot}1, 0{\cdot}2]$ and $I_2[1{\cdot}8, 1{\cdot}9]$, in which interval would you expect $f(x)$ to be more sensitive to small changes in $x$? **2**

**B2.** The following data are available for a function $f$:

| $x$ | 0·5 | 1·5 | 3·0 | 4·5 |
|---|---|---|---|---|
| $f(x)$ | 1·737 | 2·412 | 3·284 | 2·797 |

Use the cubic Lagrange interpolation formula to estimate $f(2{\cdot}5)$. **4**

**B3.** In the usual notation for forward differences of function values $f(x)$ tabulated at equally spaced values of $x$,

$$\Delta f_i = f_{i+1} - f_i$$

where $f_i = f(x_i)$ and $i = \ldots, -2, -1, 0, 1, 2, \ldots$ .

Show that $\Delta^2 f_0 = f_2 - 2f_1 + f_0$. **1**

If each value of $f_i$ is subject to an error whose magnitude is less than or equal to $\varepsilon$, determine the magnitude of the maximum possible rounding error in $\Delta^2 f_0$. **1**

Rounded values of a function $f$ are known to be $f_0 = 2{\cdot}124$, $f_1 = 2{\cdot}369$, $f_2 = 2{\cdot}618$. Obtain $\Delta^2 f_0$ and the magnitude of the maximum rounding error in $\Delta^2 f_0$. **2**

Hence state whether or not this second difference appears to be significantly different from zero. **1**

**[Turn over**

*Marks*

**B4.** The following data (accurate to the degree implied) are available for a function $f$:

| $x$ | 0 | 0·5 | 1 | 1·5 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 1·023 | 1·375 | 1·632 | 1·797 | 1·866 |

($a$) Construct a difference table of third order for the data. **2**

($b$) Using the Newton forward difference formula of degree two, and working to three decimal places, obtain an approximation to $f(0{\cdot}65)$. **3**

**B5.** Express the polynomial $f(x) = x^4 - 1{\cdot}1x^3 + 1{\cdot}7x^2 - 3{\cdot}2$ in nested form and evaluate $f(1{\cdot}3)$. **2**

It is known that the equation $f(x) = 0$ has a root very close to $x = 1{\cdot}3$ and that $f(x)$ is increasing for $1 < x < 2$. State whether the root appears to occur for $x < 1{\cdot}3$ or for $x > 1{\cdot}3$. **1**

Given that the term in $x^4$ is exact and that the other coefficients of $f(x)$ are rounded to the accuracy implied, show by considering the minimum possible value of $f(1{\cdot}3)$ that it is possible that the root may in fact be located on the other side of the point $x = 1{\cdot}3$. **2**

**B6.** Write the equations

$$\begin{aligned} 0{\cdot}3x_1 \qquad\quad &+ \quad 2x_3 &&= 8{\cdot}6 \\ 4x_1 - 0{\cdot}3x_2 \; &+ \; 0{\cdot}5x_3 &&= 6{\cdot}1 \\ 0{\cdot}5x_1 - \quad 7x_2 \; &+ \; 0{\cdot}7x_3 &&= 3{\cdot}7, \end{aligned}$$

in diagonally dominant form. Give a reason for stating that the equations are not ill-conditioned. **2**

Use the Gauss-Seidel iterative procedure with $x_1 = x_2 = x_3 = 0$ as a first approximation to solve the equations correct to two decimal places. **4**

**B7.** In the calculation using Gaussian elimination to obtain the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 2{\cdot}6 & -3{\cdot}1 & 0{\cdot}7 \\ 1{\cdot}4 & 4{\cdot}8 & 2{\cdot}3 \\ 0{\cdot}9 & -1{\cdot}9 & 3{\cdot}6 \end{pmatrix}$$

with the diagonalisation process carried out to the extent shown, the tableau of elements is:

$$\begin{pmatrix} 2{\cdot}6 & 0 & 1{\cdot}622 & 0{\cdot}742 & 0{\cdot}479 & 0 \\ 0 & 6{\cdot}469 & 1{\cdot}923 & -0{\cdot}538 & 1 & 0 \\ 0 & 0 & 3{\cdot}604 & -0{\cdot}415 & 0{\cdot}128 & 1 \end{pmatrix}.$$

Continuing to work to the same accuracy, complete the determination of the inverse of **A**, giving your answer with elements rounded to two decimal places. **6**

*Marks*

**B8.** The differential equation $\frac{dy}{dx} = \sqrt{x^2 + 2y - 1} - 1$ with $y(1) = 1$ is to be solved numerically.

(*a*) Use Euler's method with a step length of 0·1 to obtain an approximation at $x = 1{\cdot}3$ to the solution of this equation. Perform the calculations using three decimal place accuracy. **3**

State the order of the truncation order in this solution. **1**

(*b*) Use the predictor-corrector method with Euler's method as predictor and the trapezium rule as corrector to obtain a solution of this equation at $x = 1{\cdot}1$. Use one application of the corrector with a step length $h = 0{\cdot}1$ and perform the calculation using four decimal place accuracy. **4**

Comment on what this answer reveals about the accuracy of the estimate of $y(1{\cdot}1)$ obtained in part (*a*) of the question. **1**

**B9.** The following data are available for a function *f*:

| $x$ | 1 | 1·25 | 1·5 | 1·75 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 1·2690 | 1·1803 | 0·9867 | 0·6839 | 0·2749 |

Use the composite trapezium rule with two strips and four strips to obtain estimates $I_1$ and $I_2$ respectively for the integral $I = \int_1^2 f(x)\,dx$.

Perform the calculations using four decimal places. **3**

By constructing an appropriate difference table, obtain an estimate of the maximum truncation error in $I_2$. **3**

Hence state the value of $I_2$ to a suitable accuracy. **1**

If data of similar accuracy were available for the intermediate points with $x = 1{\cdot}125$, 1·375, etc, and the calculations were done with eight strips, by what factor would you expect the truncation error to be reduced? **1**

By considering appropriate Taylor series expansions for a definite integral, establish Richardson's formula to improve the accuracy of the trapezium rule by interval halving. **3**

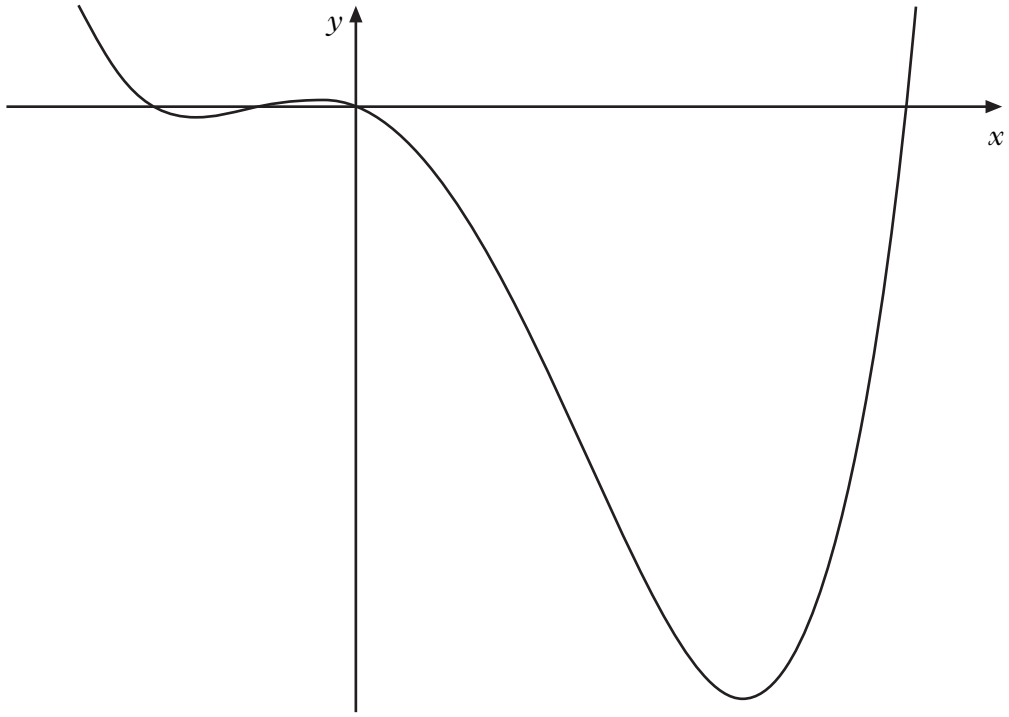
Use Richardson extrapolation to obtain an improved estimate $I_3$ for $I$ based on the values of $I_1$ and $I_2$. **1**

**[Turn over**

*Marks*

**B10.** The equation $f(x) = 0$ has a root close to $x = x_0$. By drawing a suitable graph to illustrate this situation, derive the formula for the first iteration of the Newton-Raphson method of solution $f(x) = 0$. Hence explain how the general formula is obtained. **3**

The diagram shows part of the graph of $f(x) = e^{-x} + x^4 - 2x^3 - 5x^2 - 1$ and shows that $f(x) = 0$ has four distinct real roots.




One root is known to lie in the interval [3·4, 3·6]. Use the Newton-Raphson method to determine this root correct to two decimal places. **2**

The equation has a root at $x = 0$ and another in the interval [–0·3, 0]. Use a diagram to explain why the Newton-Raphson method may be difficult to use to determine this negative root. **2**

The fourth root is given to lie in the interval [–1·1, –1]. Use three applications of the bisection method to determine a more accurate estimate of the interval in which this root lies. **3**

[*END OF SECTION B*]

**Candidates who have attempted Section B (Numerical Analysis 1 and 2) should now attempt ONE of the following**

**Section D (Mathematics 1) on *Page fifteen***

**Section E (Statistics 1) on *Pages sixteen* and *seventeen***

**Section G (Mechanics 1) on *Pages twenty* and *twenty-one*.**

*Marks*

## Section C (Mechanics 1 and 2)

**ONLY candidates doing the course Mechanics 1 and 2 and one unit chosen from Mathematics 1 (Section D), Statistics 1 (Section E) and Numerical Analysis 1 (Section F) should attempt this Section.**

**Answer all the questions.**

**Answer these questions in a separate answer book, showing clearly the section chosen.**

**Candidates should observe that $g$ m s$^{-2}$ denotes the magnitude of the acceleration due to gravity. Where appropriate, take its numerical value to be 9·8 m s$^{-2}$.**

**C1.** The position of a power sledge on a frozen lake at time $t$ seconds, relative to a rectangular coordinate system, is

$$\mathbf{r}(t) = (2t^2 - t)\mathbf{i} - (3t + 1)\mathbf{j},$$

where **i**, **j** are unit vectors in the $x$, $y$ directions respectively and distances are measured in metres.

Calculate the time at which the speed is 5 m s$^{-1}$. **4**

**C2.** At 2 pm, a ferry leaves port $O$ travelling at $25\sqrt{2}$ km/h in a north-easterly direction. At the same time, a liner is 10 km east of $O$ and travelling due north at 20 km/h. Both velocities remain constant.

(*a*) By choosing an appropriate rectangular coordinate system with origin $O$, find the position of the ferry relative to the liner at time $t$, measured in hours from 2 pm. **4**

(*b*) Calculate the distance between the ferry and the liner at 3 pm. **2**

**C3.** A piston connected to a water wheel oscillates about a point $O$ with simple harmonic motion of period $8\pi$ seconds and maximum acceleration 0·25 m s$^{-2}$.

(*a*) Calculate the amplitude of the motion. **3**

(*b*) Calculate the positions, relative to $O$, of the piston when it is moving with half its maximum speed. **4**

**C4.** A ramp consists of a rough plane inclined at angle $\theta$ to the horizontal, where $\tan\theta = \frac{5}{12}$. A box of mass $m$ kg is given a push up the line of greatest slope of the ramp, which gives the box an initial speed of $\sqrt{gL}$ m s$^{-1}$, where $L$ metres is the distance travelled before the box comes to rest.

Calculate the value of the coefficient of friction between the box and the surface of the ramp. **7**

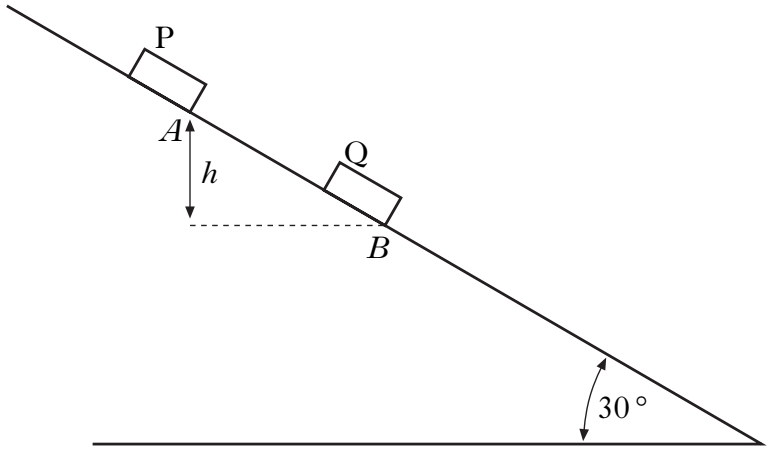
**[Turn over**

*Marks*

**C5.** An unladen helicopter of mass $M$ kilograms can hover at a constant height above the ground when the engine exerts a lift force of $P$ newtons.

The helicopter is loaded with cargo which increases its mass by 1%. When airborne, the engine now exerts a lift force 5% greater than $P$ to accelerate the helicopter vertically upwards. Calculate this vertical acceleration. **5**

**C6.** The diagram shows a ramp, inclined at 30° to the horizontal, which has a smooth section above $B$ and a rough section below $B$. Identical blocks, P and Q, each has weight $W$ newtons. Block Q is stationary at $B$, held by friction, and block P is held at rest at $A$. Block P is a vertical height of $h$ metres above block Q (where the dimensions of the blocks should be ignored).




When block P is released, it slides down the ramp colliding and coupling with block Q. The combined blocks then move down the rough section of the ramp, coming to rest at a vertical height $\frac{1}{2}h$ metres below $B$.

(i) Find, in terms of $g$ and $h$, the speed of the combined block immediately after the collision. **3**

(ii) Using the work/energy principle, show that the constant frictional force acting on the combined block whilst it is moving has magnitude $\frac{3}{2}W$ newtons. **4**

*Marks*

**C7.** A football is kicked from a point $O$ on a horizontal plane, giving the ball an initial speed $V\,\text{m}\,\text{s}^{-1}$ at an angle $\alpha$ to the horizontal. Assuming that gravity is the only force acting on the ball:

(*a*) Show that the maximum height, $H$ metres, attained by the football is given by

$$H = \frac{V^2}{2g}\sin^2\alpha.$$ **3**

(*b*) A second identical football is kicked from $O$ with the same initial speed $V\,\text{m}\,\text{s}^{-1}$ but at angle of projection $2\alpha$ to the horizontal ($2\alpha < \frac{1}{2}\pi$). The maximum height attained by this football is $h$ metres.

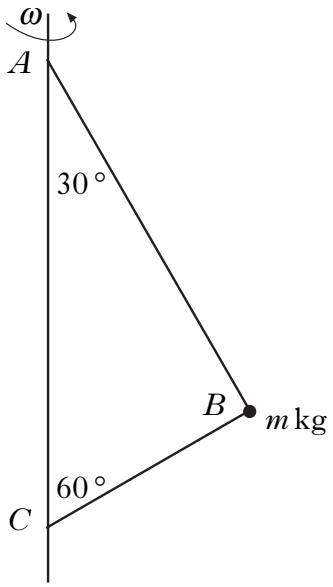
(i) Show that

$$h = 4H\left(1 - \frac{2gH}{V^2}\right).$$ **3**

[Note that $\sin 2\alpha = 2\sin\alpha\cos\alpha$.]

(ii) Given that the maximum height attained by the second football is three times that attained by the first, find the angles of projection of each of the two footballs. **4**

**C8.** A bead of mass $m$ kilograms is attached to a vertical rotating column by two strings, as shown below. String $AB$ is elastic, with natural length $L$ metres and modulus of elasticity $2mg$ newtons. The string is attached to the column at $A$ and to the bead at $B$. String $BC$ is inextensible and has length $L$ metres. The vertical column is rotating at $\omega\,\text{rad}\,\text{s}^{-1}$, such that the strings $AB$ and $BC$ are taut and remain in a vertical plane. Angles $ACB$ and $BAC$ are $60°$ and $30°$ respectively.




(*a*) Show that the tension in the string $AB$ is $2(\sqrt{3}-1)\,mg$ newtons. **4**

(*b*) Find, in terms of $m$ and $g$, an expression for the tension in the string $BC$. **3**

(*c*) Given that $L = 1$, calculate $\omega$. **4**

*Marks*

**C9.** A particle of mass $m$ kg moves in a horizontal straight line from the origin $O$ with initial velocity $U\mathbf{i}\,\mathrm{m\,s^{-1}}$, where $\mathbf{i}$ is the unit vector in the direction of motion. A resistive force $-mkv^3\mathbf{i}$ acts on the particle, where $k$ is a constant and $v\mathbf{i}$ is the velocity of the particle at time $t$ seconds measured from the start of the motion.

(i) Show that the velocity of the particle satisfies the differential equation

$$\frac{dv}{dx} = -kv^2,$$

where $x$ is the distance of the particle from $O$. **2**

Hence show that $v = \dfrac{U}{1+kUx}$. **3**

(ii) Using (i), or otherwise, show that

$$kUx^2 + 2x = 2Ut.$$ **3**

(iii) Find an expression, in terms of $k$ and $U$, for the time taken for the speed of the particle to reduce to half its initial value. **3**

[*END OF SECTION C*]

**All candidates who have attempted Section C (Mechanics 1 and 2) should now attempt ONE of the following**

**Section D (Mathematics 1) on *Page fifteen***

**Section E (Statistics 1) on *Pages sixteen* and *seventeen***

**Section F (Numerical Analysis 1) on *Pages eighteen* and *nineteen*.**

*Marks*

## Section D (Mathematics 1)

**Answer all the questions.**

**Answer these questions in a separate answer book, showing clearly the section chosen.**

**D1.** Expand $(4x - 5y)^4$ simplifying as far as possible. **4**

When $y = \frac{1}{x}$, find the term independent of $x$. **1**

**D2.** For the function defined by $y = x^2 \ln x,\ x > 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **4**

Hence show that $x\frac{d^2y}{dx^2} - \frac{dy}{dx} = kx,$ stating the value of the constant $k$. **2**

**D3.** For the following system of equations in $a$, $b$ and $c$

$$\begin{aligned} a + b - 2c &= -6 \\ 3a - b + c &= 7 \\ 2a + b - \lambda c &= -2 \end{aligned}$$

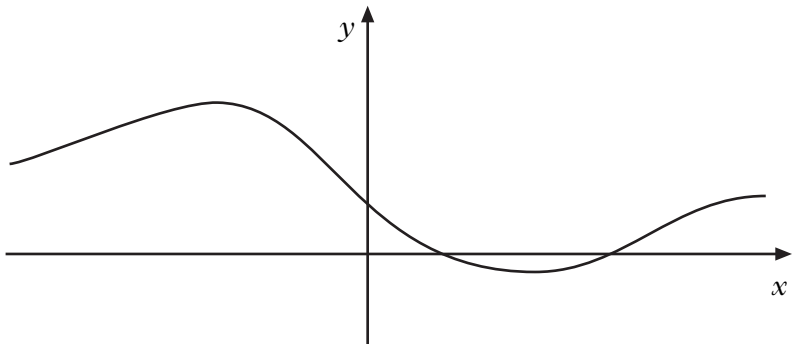
use Gaussian elimination to find

($a$) the value of $\lambda$ for which there is no solution, **3**

($b$) the values of $a$, $b$ and $c$ when $\lambda = 1$. **2**

**D4.** Use the substitution $u = x + 1$ to obtain $\int \frac{x^2 + 2}{(x+1)^2} dx$. **5**

**D5.**




The diagram shows part of the graph of $y = f(x)$ where $f(x) = \frac{(x-1)(x-4)}{x^2+4}$.

($a$) Express $f(x)$ in the form $A + \frac{Bx + C}{x^2 + 4}$ for suitable constants $A$, $B$ and $C$. **3**

($b$) Identify the asymptote of the curve. **1**

($c$) Obtain the stationary points. **3**

($d$) Evaluate the area of the finite region bounded by the curve and the $x$-axis. **4**

[*END OF SECTION D*]

*Marks*

## Section E (Statistics 1)

**Answer all the questions.**

**Answer these questions in a separate answer book, showing clearly the section chosen.**

**E1.** A discount warehouse company has branches throughout Scotland. Customers (trade and private) must be account holders and there are four times as many trade customers as private customers. In order to obtain data from customers on a variety of issues, senior management discussed the following approaches.

(*a*) Carry out telephone interviews of 80 trade and 20 private customers selected at random from the database of all customers held at head office.

(*b*) Interview 80 trade and 20 private customers leaving the Glasgow branch.

State the type of sampling involved in each approach.

State which approach is likely to yield the better estimates of parameters of the population of customers and why you rejected the other approach. **4**

**E2.** A budget airline uses Boeing 737 aircraft with 189 seats. In management discussion of seat reservation strategies, it was assumed that each passenger with a reserved seat has probability 0·002 of failing to show up for the flight. Suppose that a strategy of selling 192 seats for such flights was to be adopted.

(*a*) State the distribution of the number of passengers who fail to show up for a fully booked flight. **2**

(*b*) Obtain the probability that a fully booked flight can depart without any passengers with reserved seats having to be left behind. **3**

(*c*) Specify the distribution which would be appropriate to use if only an estimate of this probability is required. **2**

**E3.** Prior to a major maintenance of a batch chemical production plant, the mean yield was 382 kg per batch. Experience has shown that, during periods of stable production, yields vary with standard deviation 10 kg. When production was resumed, a random sample of yields (kg) was:

418 411 401 394 397.

Stating two assumptions required, obtain a 95% confidence interval for the mean yield following maintenance.

Comment on the effect of the maintenance on the mean yield. **6**

*Marks*

**E4.** Under the Weights and Measures (Packaged Goods) Regulations 1986, display of the text **500 g℮** on a packet of cereal requires the packer to ensure that:

(i) the actual contents of the packets shall not be less, on average, than 500 g;

(ii) not more than 2·5% of packets may be non-standard ie contain less than the stated content weight by more than the Tolerable Negative Error (TNE) specified.

Given that the TNE for a packet displaying **500 g℮** is 3% and that content weight is normally distributed, calculate the maximum allowable standard deviation when the mean content weight is set at 505 g. **5**

A third legal requirement is that no package may be sub-standard ie deficient by more than twice the TNE. Explain why this requirement cannot be satisfied in the above scenario. **1**

**E5.** Marine biologists studied samples of tagged salmon of both Alaskan and Canadian origin. From measurements of the growth ring diameter, $X$, of scales for the period of time the fish lived in fresh water, they established that $X$ could be modelled by N (100, $20^2$) for fish of Alaskan origin and by N (140, $20^2$) for fish of Canadian origin.

In order to predict the origin of untagged fish caught, the biologists adopted the following rule:

**Predict fish to be of Alaskan origin if $X < 120$,**
**otherwise predict fish to be of Canadian origin.**

(*a*) Calculate the probability that application of this rule would lead to a salmon of Alaskan origin being misclassified as being of Canadian origin. **3**

(*b*) State, with justification, the probability that application of the rule would lead to a salmon of Canadian origin being misclassified as being of Alaskan origin. **1**

(*c*) Given that 60% of salmon in the area are of Alaskan origin, calculate the probability that a salmon, predicted by the rule to be of Alaskan origin, is actually of Canadian origin. **5**

[*END OF SECTION E*]

**[Turn over**

*Marks*

## Section F (Numerical Analysis 1 and 2)

**Answer all the questions.**

**Answer these questions in a separate answer book, showing clearly the section chosen.**

**F1.** The polynomial $p$ is the Taylor polynomial of degree three for a function $f$ near $x = 1$.

Given the function $f(x) = \ln(2 - x)$, where $x < 2$, express $p(1 + h)$ in the form $c_0 + c_1h + c_2h^2 + c_3h^3$. **2**

Estimate the value of $\ln(1{\cdot}1)$ using this approximation, giving your answer to four decimal places. **2**

Write down the first degree Taylor polynomial for $f$ near $x = a$ where $a < 2$. Given the intervals $I_1[0{\cdot}1, 0{\cdot}2]$ and $I_2[1{\cdot}8, 1{\cdot}9]$, in which interval would you expect $f(x)$ to be more sensitive to small changes in $x$? **2**

**F2.** The following data are available for a function $f$:

| $x$ | 0·5 | 1·5 | 3·0 | 4·5 |
|---|---|---|---|---|
| $f(x)$ | 1·737 | 2·412 | 3·284 | 2·797 |

Use the cubic Lagrange interpolation formula to estimate $f(2{\cdot}5)$. **4**

**F3.** In the usual notation for forward differences of function values $f(x)$ tabulated at equally spaced values of $x$,

$$\Delta f_i = f_{i+1} - f_i$$

where $f_i = f(x_i)$ and $i = \ldots, -2, -1, 0, 1, 2, \ldots$ .

Show that $\Delta^2 f_0 = f_2 - 2f_1 + f_0$. **1**

If each value of $f_i$ is subject to an error whose magnitude is less than or equal to $\varepsilon$, determine the magnitude of the maximum possible rounding error in $\Delta^2 f_0$. **1**

Rounded values of a function $f$ are known to be $f_0 = 2{\cdot}124$, $f_1 = 2{\cdot}369$, $f_2 = 2{\cdot}618$. Obtain $\Delta^2 f_0$ and the magnitude of the maximum rounding error in $\Delta^2 f_0$. **2**

Hence state whether or not this second difference appears to be significantly different from zero. **1**

*Marks*

**F4.** The following data (accurate to the degree implied) are available for a function $f$:

| $x$ | 0 | 0·5 | 1 | 1·5 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 1·023 | 1·375 | 1·632 | 1·797 | 1·866 |

($a$) Construct a difference table of third order for the data. **2**

($b$) Using the Newton forward difference formula of degree two, and working to three decimal places, obtain an approximation to $f(0{\cdot}65)$. **3**

**F5.** The following data are available for a function $f$:

| $x$ | 1 | 1·25 | 1·5 | 1·75 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 1·2690 | 1·1803 | 0·9867 | 0·6839 | 0·2749 |

Use the composite trapezium rule with two strips and four strips to obtain estimates $I_1$ and $I_2$ respectively for the integral $I = \int_1^2 f(x)\,dx$.

Perform the calculations using four decimal places. **3**

By constructing an appropriate difference table, obtain an estimate of the maximum truncation error in $I_2$. **3**

Hence state the value of $I_2$ to a suitable accuracy. **1**

If data of similar accuracy were available for the intermediate points with $x$ = 1·125, 1·375, etc, and the calculations were done with eight strips, by what factor would you expect the truncation error to be reduced? **1**

By considering appropriate Taylor series expansions for a definite integral, establish Richardson's formula to improve the accuracy of the trapezium rule by interval halving. **3**

Use Richardson extrapolation to obtain an improved estimate $I_3$ for $I$ based on the values of $I_1$ and $I_2$. **1**

[*END OF SECTION F*]

**[Turn over**

*Marks*

## Section G (Mechanics 1)

**Answer all the questions.**

**Answer these questions in a separate answer book, showing clearly the section chosen.**

**Candidates should observe that $g$ m s$^{-2}$ denotes the magnitude of the acceleration due to gravity. Where appropriate, take its numerical value to be 9·8 m s$^{-2}$.**

**G1.** The position of a power sledge on a frozen lake at time $t$ seconds, relative to a rectangular coordinate system, is

$$\mathbf{r}(t) = (2t^2 - t)\mathbf{i} - (3t + 1)\mathbf{j},$$

where **i**, **j** are unit vectors in the $x$, $y$ directions respectively and distances are measured in metres.

Calculate the time at which the speed is 5 m s$^{-1}$. **4**

**G2.** At 2 pm, a ferry leaves port $O$ travelling at $25\sqrt{2}$ km/h in a north-easterly direction. At the same time, a liner is 10 km east of $O$ and travelling due north at 20 km/h. Both velocities remain constant.

(*a*) By choosing an appropriate rectangular coordinate system with origin $O$, find the position of the ferry relative to the liner at time $t$, measured in hours from 2 pm. **4**

(*b*) Calculate the distance between the ferry and the liner at 3 pm. **2**

**G3.** An unladen helicopter of mass $M$ kilograms can hover at a constant height above the ground when the engine exerts a lift force of $P$ newtons.

The helicopter is loaded with cargo which increases its mass by 1%. When airborne, the engine now exerts a lift force 5% greater than $P$ to accelerate the helicopter vertically upwards. Calculate this vertical acceleration. **5**

**G4.** A ramp consists of a rough plane inclined at angle $\theta$ to the horizontal, where $\tan\theta = \frac{5}{12}$. A box of mass $m$ kg is given a push up the line of greatest slope of the ramp, which gives the box an initial speed of $\sqrt{gL}$ m s$^{-1}$, where $L$ metres is the distance travelled before the box comes to rest.

Calculate the value of the coefficient of friction between the box and the surface of the ramp. **7**

*Marks*

**G5.** A football is kicked from a point $O$ on a horizontal plane, giving the ball an initial speed $V\,\text{m s}^{-1}$ at an angle $\alpha$ to the horizontal. Assuming that gravity is the only force acting on the ball:

(*a*) Show that the maximum height, $H$ metres, attained by the football is given by

$$H = \frac{V^2}{2g}\sin^2\alpha.$$ **3**

(*b*) A second identical football is kicked from $O$ with the same initial speed $V\,\text{m s}^{-1}$ but at angle of projection $2\alpha$ to the horizontal ($2\alpha < \frac{1}{2}\pi$). The maximum height attained by this football is $h$ metres.

(i) Show that

$$h = 4H\left(1 - \frac{2gH}{V^2}\right).$$ **3**

[Note that $\sin 2\alpha = 2\sin\alpha\cos\alpha$.]

(ii) Given that the maximum height attained by the second football is three times that attained by the first, find the angles of projection of each of the two footballs. **4**

[*END OF SECTION G*]

[*END OF QUESTION PAPER*]

**[BLANK PAGE]**

**[BLANK PAGE]**

**[BLANK PAGE]**