2011 Maths

Advanced Higher

Finalised Marking Instructions

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General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1. The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2. The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3. The following are not penalised:
   - working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
   - legitimate variation in numerical values / algebraic expressions.

4. Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5. Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6. Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There is one code used M. This indicates a method mark, so in question 1(a), 1M means a method mark for the product rule.
### Advanced Higher Mathematics 2011

<table>
<thead>
<tr>
<th>Marks awarded for</th>
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1. \[
\frac{13 - x}{x^2 + 4x - 5} = \frac{13 - x}{(x - 1)(x + 5)} = \frac{A}{x - 1} + \frac{B}{x + 5}
\]

\[
13 - x = A(x + 5) + B(x - 1)
\]

\[
x = 1 \Rightarrow 12 = 6A \Rightarrow A = 2
\]

\[
x = -5 \Rightarrow 18 = -6B \Rightarrow B = -3
\]

Hence

\[
\int \frac{13 - x}{x^2 + 4x - 5} \, dx = \int \frac{2}{x - 1} \, dx - \int \frac{3}{x + 5} \, dx
\]

\[
= 2 \ln |x - 1| - 3 \ln |x + 5| + c
\]

2. \[
\left(\frac{1}{2}x - 3\right)^4 = 4C_0 \left(\frac{1}{2}\right)^4 + 4C_1 \left(\frac{1}{2}\right)^3 (-3) + 4C_2 \left(\frac{1}{2}\right)^2 (-3)^2 + 4C_3 \left(\frac{1}{2}\right) (-3)^3 + 4C_4 (-3)^4
\]

\[
= \left(\frac{1}{2}\right)^4 + 4 \left(\frac{1}{2}\right)^3 (-3) + 6 \left(\frac{1}{2}\right)^2 (-3)^2 + 4 \left(\frac{1}{2}\right) (-3)^3 + (-3)^4
\]

\[
= \frac{x^4}{16} - \frac{3x^3}{2} + 27x^2 + 54x + 81
\]

3. (a) Method 1

\[
y + e^y = x^2
\]

\[
\frac{dy}{dx} + e^y \frac{dy}{dx} = 2x
\]

\[
\frac{dy}{dx} (1 + e^y) = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{1 + e^y}
\]

Method 2

\[
\ln (y + e^y) = 2 \ln x
\]

\[
\frac{1 + e^y}{y + e^y} \frac{dy}{dx} = \frac{2}{x}
\]

\[
\frac{dy}{dx} = \frac{2(y + e^y)}{x(1 + e^y)}
\]

Method 3

\[
y + e^y = x^2 \Rightarrow e^y = x^2 - y \Rightarrow y = \ln (x^2 - y)
\]

\[
\frac{dy}{dx} = \frac{2x - \frac{dy}{dx}}{x^2 - y}
\]

\[
\frac{dy}{dx} = \frac{2x}{x^2 - y + 1}
\]
3. **(b) Method 1**

\[ f(x) = \sin x \cos^3 x \]

\[ f'(x) = \cos^4 x + \sin x (-3 \cos^2 x \sin x) \]

\[ = \cos^4 x - 3 \cos^2 x \sin^2 x \]

**Method 2**

\[ f(x) = \sin x \cos^3 x \]

\[ \ln(f(x)) = \ln \sin x + \ln (\cos^3 x) \]

\[ f'(x) = \frac{\cos x}{\sin x} - 3 \frac{\cos^2 x \sin x}{\cos x} \]

\[ = \frac{\cos x}{\sin x} - 3 \frac{\sin x}{\cos x} \]

\[ f'(x) = \left( \frac{\cos x}{\sin x} - 3 \frac{\sin x}{\cos x} \right) \sin x \cos^3 x \]

\[ = \cos^4 x - 3 \sin^2 x \cos^2 x \]

4. **(a)** Singular when the determinant is 0.

\[ 1 \det \begin{pmatrix} 0 & 2 \\ \lambda & 6 \end{pmatrix} - 2 \det \begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix} + (-1) \det \begin{pmatrix} 3 & 0 \\ -1 & \lambda \end{pmatrix} = 0 \]

\[ -2\lambda - 2 (18 + 2) - 1 (3\lambda - 0) = 0 \]

\[ -5\lambda - 40 = 0 \text{ when } \lambda = -8 \]

**(b)** From \( A, A' = \begin{pmatrix} 2 & 3\alpha + 2\beta & -1 \\ 2\alpha - \beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix} \)

Hence \( 2\alpha - \beta = -1 \) and \( 3\alpha + 2\beta = -5 \).

\[ 4\alpha - 2\beta = -2 \]

\[ 3\alpha + 2\beta = -5 \]

\[ 7\alpha = -7 \Rightarrow \alpha = -1 \text{ and } \beta = -1. \]
5. (6) Let \( f(x) = (1 + x)^{\frac{1}{2}} \), then
\[
\begin{align*}
    f(x) &= (1 + x)^{\frac{1}{2}} \Rightarrow f(0) = 1 \\
    f'(x) &= \frac{1}{2}(1 + x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2} \\
    f''(x) &= -\frac{1}{4}(1 + x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4} \\
    f'''(x) &= \frac{3}{8}(1 + x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}
\end{align*}
\]
Hence
\[
\sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{4} \times \frac{x^2}{2} + \frac{3}{8} \times \frac{x^3}{6} - \ldots
\]
and replacing \( x \) by \( x^2 \) gives
\[
\sqrt{1 + x^2} = 1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{16} - \ldots
\]
Thus
\[
\sqrt{(1 + x)(1 + x^2)} = \left(1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \ldots\right)\left(1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{16} - \ldots\right)
\]
\[
= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots
\]

6. (4) Reflect in the line \( y = x \) to get
\[
\begin{align*}
    \text{Position:} & \quad (a, 0) \\
    \text{Coordinates:} & \quad (0, -1)
\end{align*}
\]
Now apply the modulus function
\[
\begin{align*}
    \text{Shape:} & \quad (0, 1) \\
    \text{Coordinates:} & \quad (a, 0)
\end{align*}
\]
7. Method 1

\[ y = \frac{e^{\sin x} (2 + x)^3}{\sqrt{1 - x}} \]

\[ \Rightarrow \ln y = \ln \left( e^{\sin x} (2 + x)^3 \right) - \ln (\sqrt{1 - x}) \]

\[ = \sin x + 3 \ln (2 + x) - \frac{1}{2} \ln (1 - x) \]

\[ \Rightarrow \frac{1}{y} \frac{dy}{dx} = \cos x + \frac{3}{2 + x} + \frac{1}{2(1 - x)} \]

\[ \frac{dy}{dx} = y \left( \cos x + \frac{3}{2 + x} + \frac{1}{2(1 - x)} \right) \]

When \( x = 0, y = 8 \Rightarrow \)

gradient = \( 8 \left( 1 + \frac{3}{2} + \frac{1}{2} \right) = 24 \)

Method 2

\[ y = \frac{e^{\sin x} (2 + x)^3}{\sqrt{1 - x}} \Rightarrow \frac{dy}{dx} = \frac{4}{dx} \left( e^{\sin x} (2 + x)^3 \right) \sqrt{1 - x} - e^{\sin x} (2 + x)^3 \left( -\frac{1}{2} \right) \]

\[ = \left[ \cos x e^{\sin x} (2 + x)^3 + 3e^{\sin x} (2 + x)^2 \right] (1 - x) \]

\[ + \frac{e^{\sin x} (2 + x)^3}{2(1 - x)^{3/2}} \]

When \( x = 0, \)

gradient = \( \frac{(2^3 + 3 \times 2^2)}{1} + \frac{2^3}{2} = 20 + 4 = 24 \)

Method 3

\[ y = \frac{e^{\sin x} (2 + x)^3}{\sqrt{1 - x}} \]

\[ y \sqrt{1 - x} = e^{\sin x} (2 + x)^3 \]

\[ \sqrt{1-x} \frac{dy}{dx} - \frac{y}{2} (1-x)^{-1/2} = \cos x e^{\sin x} (2 + x)^3 + 3e^{\sin x} (2 + x)^2 \]

when \( x = 0, y = e^{\sin 1} = 8. \) This leads to \( \frac{dy}{dx} = 24 \)
8. \[
\sum_{r=1}^{n} r^3 - \left( \sum_{r=1}^{n} r \right)^2 = \frac{n^2(n+1)^2}{4} - \left( \frac{n(n+1)}{2} \right)^2 = 0
\]

\[
\sum_{r=1}^{n} r^3 + \left( \sum_{r=1}^{n} r \right)^2 = \frac{n^2(n+1)^2}{4} + \left( \frac{n(n+1)}{2} \right)^2
\]

\[
= \frac{n^2(n+1)^2}{4} + \frac{n^2(n+1)^2}{4}
\]

\[
= \frac{n^2(n+1)^2}{2}
\]

**Marks awarded for**

(4) 1

9. **Method 1**

\[
\frac{dy}{dx} = 3(1 + y)\sqrt{1 + x}
\]

\[
\int \frac{dy}{1 + y} = 3 \int (1 + x)^{\frac{3}{2}} dx
\]

M1 1

for LHS

for term in \(x\)

for the constant

\[
\ln (1 + y) = 2 (1 + x)^{\frac{3}{2}} + c
\]

\[
y = \exp \left( 2 (1 + x)^{\frac{3}{2}} + c \right) - 1.
\]

\[
y = A \exp \left( 2 (1 + x)^{\frac{3}{2}} \right) - 1.
\]

**Method 2**

\[
\frac{dy}{dx} - 3\sqrt{1 + x} y = 3\sqrt{1 + x}
\]

Integrating Factor

\[
\exp \left( -3 \int \sqrt{1 + x} dx \right) = \exp \left( -2 (1 + x)^{\frac{3}{2}} \right)
\]

\[
\frac{dy}{dx} (y \exp \left( -2 (1 + x)^{\frac{3}{2}} \right)) =
\]

\[
3\sqrt{1 + x} (\exp \left( -2 (1 + x)^{\frac{3}{2}} \right))
\]

\[
y \left( \exp \left( -2 (1 + x)^{\frac{3}{2}} \right) \right) =
\]

\[
- \int \left( -3\sqrt{1 + x} \exp \left( -2 (1 + x)^{\frac{3}{2}} \right) \right) dx
\]

\[
= - \exp \left( -2 (1 + x)^{\frac{3}{2}} + c \right)
\]

\[
y = -1 + c \exp \left( 2 (1 + x)^{\frac{3}{2}} \right)
\]

**Marks awarded for**

(5) 1

10. Let \( z = x + iy \), so

\[
z - 1 = (x - 1) + iy.
\]

\[
|z - 1|^2 = (x - 1)^2 + y^2 = 9.
\]

The locus is the circle with centre \((1, 0)\) and radius 3.

Can subsume the first two marks.

for circle

for shading or other indication

**Marks awarded for**

(5) 1
11. (a) \[ \int_0^{\pi/4} (\sec x - x) (\sec x + x) \, dx = \int_0^{\pi/4} (\sec^2 x - x^2) \, dx \]

\[ = \left[ \tan x - \frac{x^3}{3} \right]_0^{\pi/4} \]

\[ = \left[ 1 - \frac{\pi^3}{364} \right] - [0] \]

\[ = 1 - \frac{\pi^3}{192}. \]

(b) Method 1

Let \( u = 7x^2 \), then \( du = 14x \, dx \).

\[ \int \frac{x}{\sqrt{1 - 49x^2}} \, dx = \frac{1}{14} \int \frac{du}{\sqrt{1 - u^2}} \]

\[ = \frac{1}{14} \sin^{-1} u + c \]

\[ = \frac{1}{14} \sin^{-1} 7x^2 + c \]

Method 2

\[ \int \frac{x}{\sqrt{1 - 49x^2}} \, dx = \frac{1}{14} \int \frac{14x \, dx}{\sqrt{1 - (7x^2)^2}} \]

\[ = \frac{1}{14} \sin^{-1} 7x^2 + c \]

12. (5)

For \( n = 2 \), \( 8^2 + 3^0 = 64 + 1 = 65 \).

Assume true for \( k \), i.e. that \( 8^k + 3^{k-2} \) is divisible by 5, i.e. can be expressed as \( 5p \) for an integer \( p \). 

Now consider \( 8^{k+1} + 3^{k-1} \)

\[ = 8 \times 8^k + 3^{k-1} \]

\[ = 8 \times (5p - 3^{k-2}) + 3^{k-1} \]

\[ = 40p - 3^{k-2}(8 - 3) \]

\[ = 5(8p - 3^{k-2}) \] which is divisible by 5.

So, since it is true for \( n = 2 \), it is true for all \( n \geq 2 \).
13. **Method 1**

Let $d$ be the common difference. Then

\[ u_3 = 1 = a + 2d \quad \text{and} \quad u_2 = \frac{1}{a} = a + d \]

1

\[ 1 = a + 2\left(\frac{1}{a} - a\right) \]

1

\[ a = a^2 + 2 - 2a^2 \]

1

\[ a^2 + a - 2 = 0 \]

1

\[ (a + 2)(a - 1) = 0 \Rightarrow a = -2 \text{ since } a < 0. \]

1

\[ a = -2 \text{ gives } 2d = 3 \text{ and hence } d = \frac{3}{2}. \]

1

**Method 2**

\[ u_1 = a, u_2 = \frac{1}{a}, u_3 = 1 \]

M1

\[ \Rightarrow \frac{1}{a} - a = 1 - \frac{1}{a} \]

1

\[ \Rightarrow 1 - a^2 = a - 1 \]

1

\[ \Rightarrow a^2 + a - 2 = 0 \]

1

\[ (a + 2)(a - 1) = 0 \Rightarrow a = -2 \text{ since } a < 0. \]

1

\[ d = u_3 - u_2 = 1 - \frac{1}{a} = \frac{3}{2} \]

1

\[ S_n = \frac{n}{2} [2a + (n - 1)d] \]

\[ = \frac{n}{2} \left[ -4 + \frac{3}{2}n - \frac{3}{2} \right] \]

1

\[ = \frac{1}{4} [3n^2 - 11n] \]

\[ \therefore 3n^2 - 11n > 4000 \]

1

\[ n^2 - \frac{11}{3} n > \frac{4000}{3} \]

\[ \left( n - \frac{11}{6} \right)^2 > \frac{48000}{36} + \frac{121}{36} = \frac{48121}{36} \]

\[ n - \frac{11}{6} > \sqrt{\frac{48121}{36}} \]

\[ n > \frac{\sqrt{48121} + 11}{6} \approx 38.39 \]

So the least value of $n$ is 39. 

1 for value

1 for suitable justification
14. Auxiliary equation

\( m^2 - m - 2 = 0 \) \hspace{1cm} 1

\((m - 2)(m + 1) = 0 \)

\( m = -1 \) or \( 2 \) \hspace{1cm} 1

Complementary function is: \( y = Ae^{-x} + Be^{2x} \) \hspace{1cm} 1

The particular integral has the form \( y = Ce^x + D \) \hspace{1cm} 1

\( y = Ce^x + D \Rightarrow \frac{dy}{dx} = Ce^x \)

\( \Rightarrow \frac{d^2y}{dx^2} = Ce^x \) \hspace{1cm} 1

Hence we need:

\( \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x + 12 \)

\([Ce^x] - [Ce^x] - 2[Ce^x + D] = e^x + 12 \)

\(-2Ce^x - 2D = e^x + 12 \) \hspace{1cm} 1

Hence \( C = -\frac{1}{2} \) and \( D = -6 \).

So the General Solution is

\( y = Ae^{-x} + Be^{2x} - \frac{1}{2}e^x - 6. \) \hspace{1cm} 1

\( x = 0 \) and \( y = -\frac{3}{2} \Rightarrow \)

\( A + B - \frac{1}{2} - 6 = -\frac{3}{2} \) \hspace{1cm} 1

Setting up the equations

\( x = 0 \) and \( \frac{dy}{dx} = \frac{1}{2} \Rightarrow \)

\( -A + 2B - \frac{1}{2} = \frac{1}{2} \)

\( 3B - 7 = 1 \Rightarrow B = 2 \Rightarrow A = 3 \) \hspace{1cm} 1

So the particular solution is

\( y = 3e^{-x} + 2e^{2x} - \frac{1}{2}e^x - 6. \) \hspace{1cm} 1
15. (a) In terms of a parameter \( s \), \( L_1 \) is given by
\[
 x = 1 + ks, \quad y = -s, \quad z = -3 + s
\]
In terms of a parameter \( t \), \( L_2 \) is given by
\[
 x = 4 + t, \quad y = -3 + t, \quad z = -3 + 2t
\]
Equating the \( y \) coordinates
and equating the \( z \) coordinates:
\[
\begin{align*}
-s &= -3 + t \\
-3 + s &= -3 + 2t
\end{align*}
\]
Adding these
\[
-3 = -6 + 3t
\]
\[
\Rightarrow t = 1 \Rightarrow s = 2.
\]
From the \( x \) coordinates
\[
1 + ks = 4 + t
\]
Using the values of \( s \) and \( t \)
\[
1 + 2k = 5 \Rightarrow k = 2
\]
The point of intersection is: \((5, -2, -1)\).

(b) \( L_1 \) has direction \( 2\mathbf{i} - \mathbf{j} + \mathbf{k} \).
\( L_2 \) has direction \( \mathbf{i} + \mathbf{j} + 2\mathbf{k} \).
For both directions.

Let the angle between \( L_1 \) and \( L_2 \) be \( \theta \), then
\[
\cos \theta = \frac{(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{\|2\mathbf{i} - \mathbf{j} + \mathbf{k}\| \|\mathbf{i} + \mathbf{j} + 2\mathbf{k}\|}
\]
\[
= \frac{2 - 1 + 2}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}
\]
\[
\theta = 60^\circ
\]
The angle between \( L_1 \) and \( L_2 \) is \( 60^\circ \).
16. (a) $I_n = \int_0^1 \frac{1}{(1 + x^2)^n} \, dx$
\[= \int_0^1 1 \times (1 + x^2)^{-n} \, dx \]
\[= \left[ (1 + x^2)^{-n} \right]_0^1 + \int_0^1 (2nx(1 + x^2)^{-n-1}) \, dx \]
\[= \left[ x(1 + x^2)^{-n} \right]_0^1 + \int_0^1 2nx^2(1 + x^2)^{-n-1} \, dx \]
\[= \frac{1}{2^n} - 0 + 2n \int_0^1 x^2 (1 + x^2)^{-n-1} \, dx \]
\[= \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(1 + x^2)^{n+1}} \, dx. \]

(b) $\frac{A}{(1 + x^2)^n} + \frac{B}{(1 + x^2)^{n+1}} = \frac{x^2}{(1 + x^2)^{n+1}}$  
\[\Rightarrow A(1 + x^2) + B = x^2 \]
\[\Rightarrow A = 1, B = -1 \]
\[\Rightarrow I_n = \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(1 + x^2)^{n+1}} \, dx. \]
\[= \frac{1}{2^n} + 2n \int_0^1 \frac{1}{(1 + x^2)^n} \, dx + 2n \int_0^1 \frac{-1}{(1 + x^2)^{n+1}} \, dx \]
\[= \frac{1}{2^n} + 2nI_n - 2nI_{n+1} \]
\[2nI_{n+1} = \frac{1}{2^n} + (2n - 1)I_n \]
\[I_{n+1} = \frac{1}{n \times 2^{n+1}} + \left( \frac{2n - 1}{2n} \right) I_n \]

(c) $\int_0^1 \frac{1}{(1 + x^2)^3} \, dx = I_3$
\[= \frac{1}{16} + \frac{3}{4}I_2 \]
\[= \frac{1}{16} + \frac{3}{4} \left( \frac{1}{4} + \frac{1}{2}I_1 \right) \]
\[= \frac{1}{4} + \frac{3}{8} \int_0^1 \frac{1}{1 + x^2} \, dx \]
\[= \frac{1}{4} + \frac{3}{8} \left[ \tan^{-1} x \right]_0^1 \]
\[= \frac{1}{4} + \frac{3\pi}{8} = \frac{1}{4} + \frac{3\pi}{32}. \]

END OF SOLUTIONS