19 / 8 / 17

Force, Energy and Periodic Motion - Lesson 7

Hooke's Law

LI

- Know and apply Hooke's Law.

SC

- Algebra.
- Free Body Diagrams (FBDs).

The natural length of a spring is its unextended or uncompressed length (its length in a 'natural state')

The spring tension is the restoring force tending to pull/push the spring back to its natural state after it has been extended/compressed

Hooke's Law states that the spring tension is directly proportional to the extension/compression of the spring from its equilibrium position (natural state) and acts in a direction opposite to that of the extension/compression

A Hookean object is one that obeys Hooke's Law

Consider a spring of natural length L that is extended a distance x from its equilibrium position :

L

$L + x$

Taking the origin to be at the leftmost position of the spring, taking right as positive, the spring tension in the extended spring to be T, and the proportionality constant k (the spring constant/stiffness constant), Hooke's Law is,

$$\underline{T} = -k\,\underline{x}$$

$$\therefore \quad T\,\underline{i} = -k\,(x\,\underline{i})$$

$$\Rightarrow \quad \underline{T = -k\,x}$$

Defining the elastic modulus (aka modulus of elasticity aka modulus) λ by $\lambda = k\,L$, Hooke's Law can be written as,

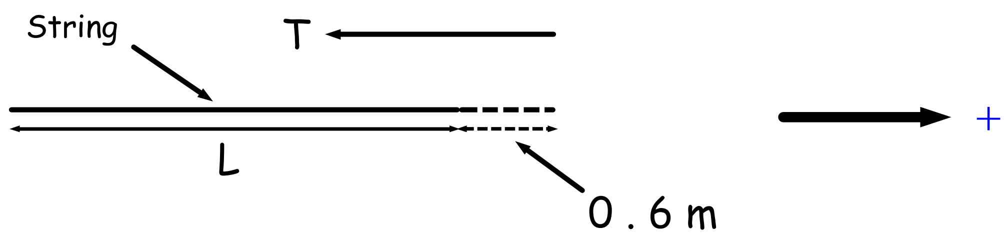
$$\boxed{T = -\frac{\lambda}{L}\,x}$$

This equation is also true for the case of a compressed spring. Hooke's Law also applies to elastic strings; for a single elastic string, there is only extension. In cases where strings are attached, there may be compression.

Example 1

An elastic string has natural length 4 m and modulus 16 N.

Find the tension in the string when the extension is 60 cm.



$$T = -\frac{\lambda}{L}\,x$$

$$\therefore \quad T = -\frac{16}{4}(0.6)$$

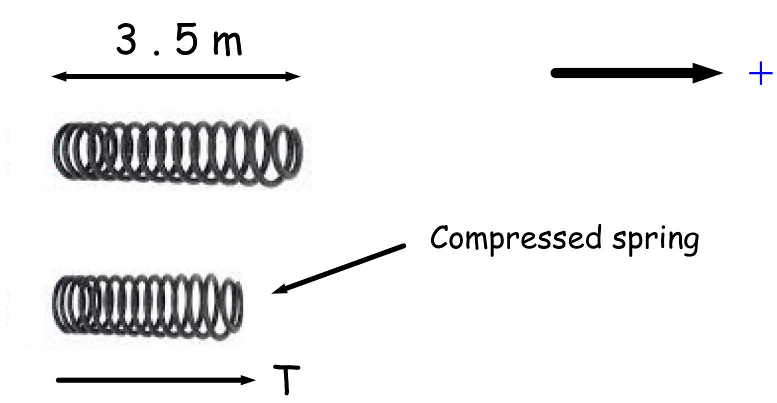
$$\Rightarrow \quad \underline{T = -2.4\text{ N}}$$

Tension is 2 . 4 N to the left (or, opposite to extension)

Example 2

A spring is of natural length 3 . 5 m and modulus 35 N.

If the thrust in the spring when it is compressed is 27 N, find the length of the compressed spring.



$$T = -\frac{\lambda}{L}\,x$$

$$\therefore \quad 27 = -\frac{35}{3.5}\,x$$

$$\Rightarrow \quad 27 = -10\,x$$

$$\Rightarrow \quad \underline{x = -2.7 \text{ m}}$$

This is the compression (not the compressed length) to the left. The compressed length $L_{COMP.}$ is the natural length minus the size of this compression. So,

$$L_{COMP.} = L - |x|$$

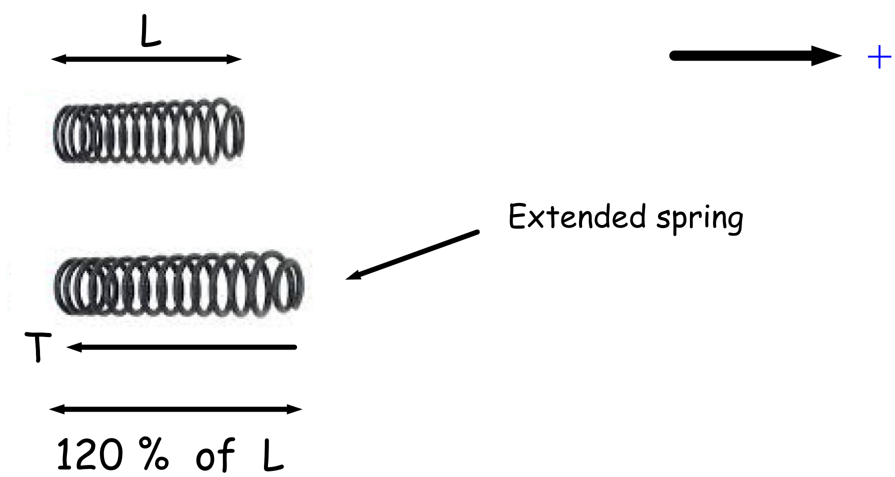
$$\therefore \quad L_{COMP.} = 3.5 - 2.7$$

$$\Rightarrow \quad \boxed{L_{COMP.} = 0.8 \text{ m}}$$

Example 3

When the length of a spring is 120 % of its unstretched length, the thrust in the spring is 8 N.

Find the modulus of the spring.



The extension is,

$$x = 1.2L - L$$

$$\Rightarrow \quad \underline{x = 0.2L}$$

$$T = -\frac{\lambda}{L}x$$

$$\therefore \quad -8 = -\frac{\lambda}{L}(0.2L)$$

$$\Rightarrow \quad -8 = -0.2\lambda$$

$$\Rightarrow \quad \boxed{\lambda = 40\text{ N}}$$

Example 4

When an elastic string is stretched to a length of 22 cm, the tension in the string is 17 N.

If the modulus of the string is 51 N, find the string's natural length.

The extension is,

$$\underline{x = 0.22 - L}$$

$$T = -\frac{\lambda}{L}x$$

$$\therefore \quad -17 = -\frac{51}{L}(0.22 - L)$$

$$\Rightarrow \quad L = 3(0.22 - L)$$

$$\Rightarrow \quad L = 0.66 - 3L$$

$$\Rightarrow \quad 4L = 0.66$$

$$\Rightarrow \quad \boxed{L = 0.165 \text{ m}}$$

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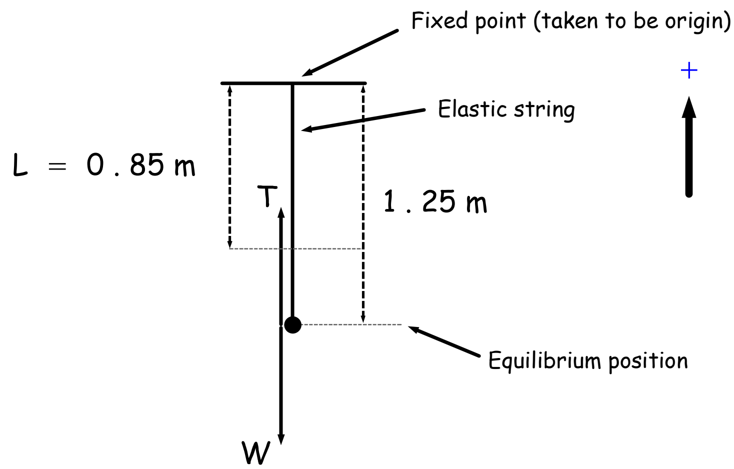
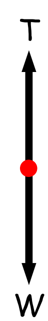
Example 5

A light, elastic string of natural length 85 cm has one end fixed with a mass of eight thousand grams freely suspended from the other end.

Show that, if the total length of the string in the equilibrium position is 1 . 25 m, the modulus of elasticity of the string is 17 g N.



$$x = -1.25 - (-0.85) \Rightarrow \underline{x = -0.4 \text{ m}}$$



In equilibrium, the restoring force is balanced by the weight. So,

$$T = W$$

$$\therefore \quad -\frac{\lambda}{L}x = mg$$

$$\Rightarrow \quad \lambda = -\frac{mgL}{x}$$

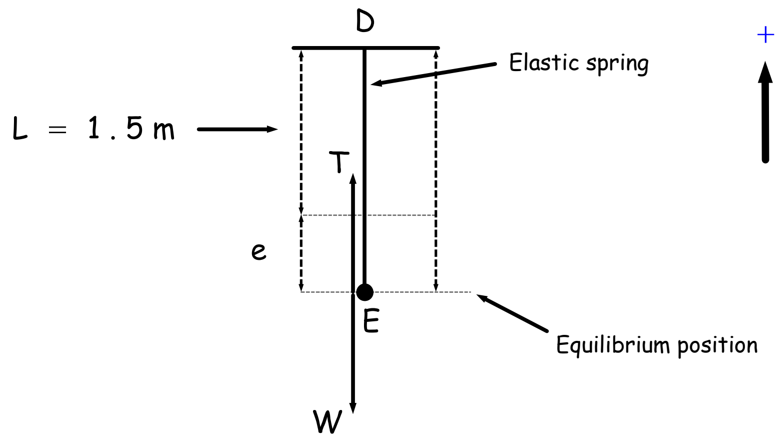
$$\therefore \quad \lambda = -\frac{8(0.85)g}{-0.4}$$

$$\Rightarrow \quad \boxed{\lambda = 17g}$$

Example 6

A light, elastic spring has its upper end D fixed and a body of mass 0 . 6 kg attached to its other end E. If the modulus of the spring is 9g/2 N and its natural length is 1 . 5 m, find the extension e of the spring when the body hangs in equilibrium.

The end E of the spring is pulled vertically downwards to F, where EF = 10 cm. Find the initial acceleration of the body when released from position F.



In equilibrium, the restoring force is balanced by the weight. So,

$$T = W$$

$$\therefore \quad -\frac{\lambda}{L}e = mg$$

$$\Rightarrow \quad e = -\frac{mgL}{\lambda}$$

$$\therefore \quad e = -\frac{(0.6)(1.5)g}{9g/2}$$

$$\Rightarrow \quad \underline{e = -0.2}$$

$$\boxed{\text{Extension is } 0.2 \text{ m downwards}}$$

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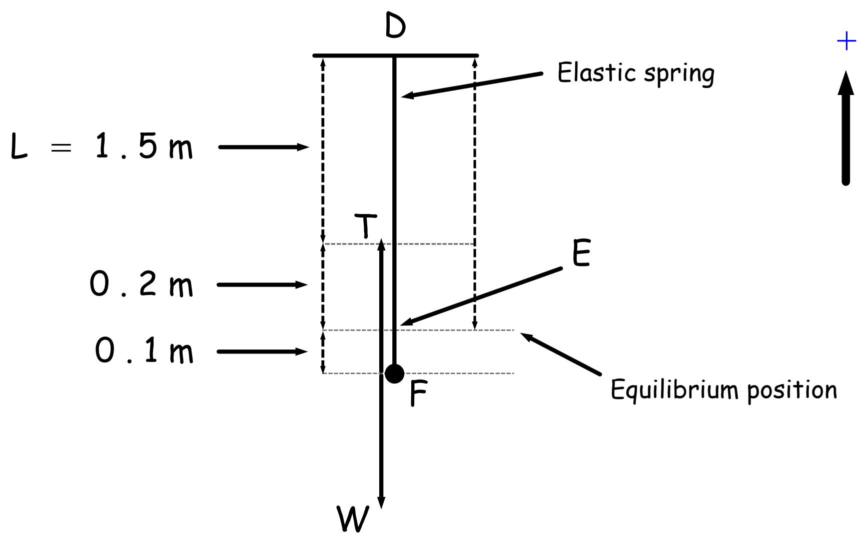
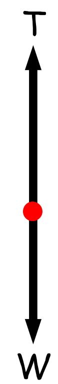


$$x = -0.2 + (-0.1) \Rightarrow \underline{x = -0.3\text{ m}}$$



Net force F_{NET} is upwards

When released from position F, the net force is upwards. So,

$$F_{NET} = T - W$$

$$\therefore \quad ma = -\frac{\lambda}{L}x - mg$$

$$\therefore \quad 0.6a = -\frac{(4.5)(9.8)}{1.5}(-0.3) - 0.6(9.8)$$

$$\Rightarrow \quad 0.6a = 8.82 - 5.88$$

$$\Rightarrow \quad 0.6a = 2.94$$

$$\Rightarrow \quad \underline{a = 4.9\text{ m s}^{-2}}$$

$\boxed{\text{Acceleration is } 4.9\text{ m s}^{-2} \text{ upwards}}$

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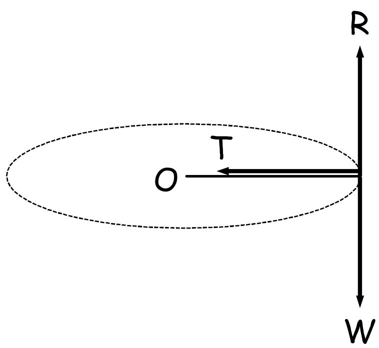
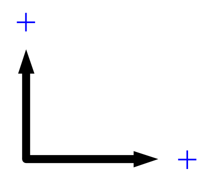
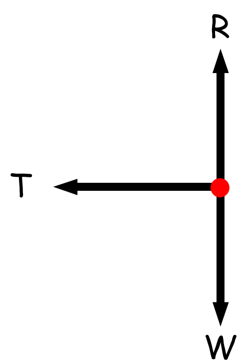
Example 7

A body lies on a smooth, horizontal surface and is connected to a point O on the surface by a light, elastic string of natural length 50 cm and modulus 70 N. When the body moves in a horizontal circular path about O with a constant speed of $3.5\ \text{m s}^{-1}$, the extension in the string is 20 cm.

Determine the mass m of the body.







Net force F_{NET} is inwards

Equating vertically gives $R = W = mg$, which doesn't reveal anything. Horizontally, the central force is provided by the string's tension. So,

$$-\frac{mv^2}{r} = -\frac{\lambda}{L}x$$

$$\Rightarrow \quad m = \frac{\lambda x r}{L v^2}$$

$$\therefore \quad m = \frac{70\,(0.2)\,(0.7)}{(0.5)\,(3.5)^2}$$

$$\Rightarrow \quad \boxed{m = 1.6\ \text{kg}}$$

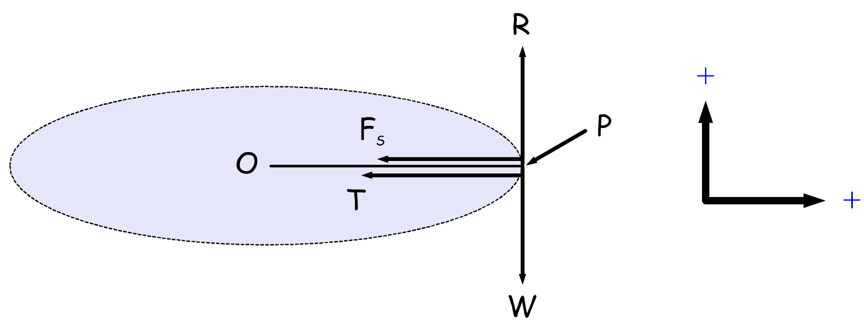
Example 8

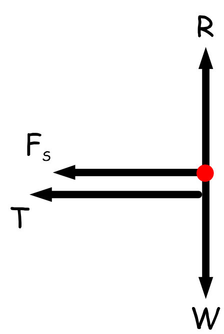
A rough disc rotates in a horizontal plane with a constant angular velocity ω rad s^{-1} about a fixed vertical axis through the centre O of the disc. A particle of mass m kg lies at a point P on the disc and is attached to the axis by a light, elastic string OP of natural length L m and elastic modulus $3mg$ N. The particle is at a distance of $5L/4$ m from O and the coefficient of static friction between P and the disc is $2/7$.

Find the range of values of ω (in terms of g and L) for which the particle remains stationary on the disc.

There is a range of values for ω as there are two cases to consider : motion to just prevent slipping outwards and motion to just prevent being pulled in towards the centre.

Preventing slipping out :





The vertical forces are balanced ($R = W = mg$) and the centripetal force is provided by the restoring force T and F_s. To be clear, write the forces, with the above directions, as,

$$\underline{F}_c = -mr\omega^2\,\underline{i}$$

$$\underline{T} = T\,\underline{i} = -\frac{\lambda}{L}x\,\underline{i}$$

$$\underline{F}_s = -F_s\,\underline{i}$$

Then,

$$\underline{\mathbf{F}}_c = \underline{\mathbf{T}} + \underline{\mathbf{F}}_s$$

$$\therefore \quad -mr\omega^2\,\underline{\mathbf{i}} = -\frac{\lambda}{L}x\,\underline{\mathbf{i}} - F_s\,\underline{\mathbf{i}}$$

$$\Rightarrow \quad mr\omega^2 = \frac{\lambda}{L}x + F_s$$

$$\Rightarrow \quad mr\omega^2 \leq \frac{\lambda}{L}x + \mu_s mg$$

$$\therefore \quad \frac{5mL}{4}\omega^2 \leq \frac{3mgL}{4L} + \frac{2mg}{7}$$

$$\Rightarrow \quad \frac{5L}{4}\omega^2 \leq \frac{3g}{4} + \frac{2g}{7}$$

$$\Rightarrow \quad \frac{5L}{4}\omega^2 \leq \frac{29g}{28}$$

$$\Rightarrow \quad \underline{\omega \leq \sqrt{\frac{29g}{35L}}}$$

Preventing pulling in :

This time, the friction force is outwards. So, $\underline{\mathbf{F}}_s = F_s \underline{\mathbf{i}}$. Thus,

$$m r \omega^2 = \frac{\lambda}{L} x - F_s$$

$$\Rightarrow \quad m r \omega^2 \geq \frac{\lambda}{L} x - \mu_s m g$$

$$\therefore \quad \frac{5 m L}{4} \omega^2 \geq \frac{3 m g L}{4 L} - \frac{2 m g}{7}$$

$$\Rightarrow \quad \frac{5 L}{4} \omega^2 \geq \frac{3 g}{4} - \frac{2 g}{7}$$

$$\Rightarrow \quad \frac{5 L}{4} \omega^2 \geq \frac{13 g}{28}$$

$$\Rightarrow \quad \underline{\omega \geq \sqrt{\frac{13 g}{35 L}}}$$

So, required range of values for ω is,

$$\boxed{\sqrt{\frac{13 g}{35 L}} \leq \omega \leq \sqrt{\frac{29 g}{35 L}}}$$

Blue Book

- pg. 363-4 Ex. 15 A Q 1 - 12, 14, 17, 18 and 20.