## $5 / 8 / 17$

Force, Energy and Periodic Motion - Lesson 6

## Simple Harmonic Motion (SHM)

## LI

- Know and use the formulae for SHM.

SC

- Algebra and calculus.

Simple Harmonic Motion (SHM) is periodic motion of an object with no damping forces and no forced motion such that the force acting on the object is proportional to the object's displacement and acts in a direction opposite to this displacement

## A Simple Harmonic Oscillator (SHO) is an object that performs SHM

A restoring force is a force that tends to bring a physical system back to an equilibrium position

The force in SHM is a restoring force. More specifically, consider an object of mass $m$ moving along the $x$ - axis (taking right as positive) with displacement $\underline{x}$ from an origin $O$ :


The restoring force is, where $k$ is a positive constant,

$$
\begin{array}{rlrl} 
& & \underline{\underline{F}}_{\text {RES }} & =-k \underline{x} \\
& \therefore & F_{\text {RES }} \underline{\underline{i}} & =-k \times \underline{\mathbf{i}} \\
\Rightarrow & & m a \underline{\mathbf{i}} & =-k \times \underline{\mathbf{i}} \\
& \therefore & m a & =-k x \\
& & & a=-(k / m) x \\
& \therefore & a=\ddot{x}=-\omega^{2} x \\
& & & \left(\omega^{2}=\frac{k}{m}\right)
\end{array}
$$

The above equation is what characterises SHM and gives the acceleration in terms of displacement. The restoring force causes the object to continually oscillate between M and N .

The equation $\ddot{x}=-\omega^{2} x$ is a second-order ODE with general solution that can be written as,

$$
x(t)=A \sin (\omega t+\theta)
$$

The period of motion of a SHO is $T=\frac{2 \pi}{\omega}$
Hence, $w>0$

We analyse when a SHO has maximum and minimum values for distance, speed and magnitude of acceleration.

For convenience, we gather together the first three time derivatives of $x(\dagger)$ (in Euler notation):

$$
\begin{aligned}
x & =A \sin (\omega t+\theta) \\
D x & =\omega A \cos (\omega t+\theta)=v \\
D^{2} x & =-\omega^{2} A \sin (\omega t+\theta)=a \\
D^{3} x & =-\omega^{3} A \cos (\omega t+\theta)
\end{aligned}
$$

Also note that,

$$
\begin{aligned}
& \sin (\omega t+\theta)=0 \text { when } \omega t+\theta=n \pi(n \in \mathbb{Z}) \\
& \cos (\omega t+\theta)=0 \text { when } \omega t+\theta=(2 n+1) \pi / 2(n \in \mathbb{Z})
\end{aligned}
$$

## Displacement

$$
x=A \sin (\omega t+\theta)
$$

As the maximum value of sine is 1 and the minimum value of sine is -1 , the displacement of the object from the equilibrium position varies from $x=-A$ to $x=A$.

## The amplitude (of motion) of a SHO is A

The maximum distance occurs when the first derivative of $x$ is 0 , i.e. $\omega A \cos (\omega t+\theta)=0 \Rightarrow \omega t+\theta=(2 n+1) \pi / 2(n \in \mathbb{Z})$.

Obviously the minimum distance is 0 and occurs when $\sin (\omega \dagger+\theta)=0 \Rightarrow \omega \dagger+\theta=n \pi(n \in \mathbb{Z})$. Hence,


Similiar analyses can be performed for velocity and acceleration. Velocity


Occurs at endpoints $M$ and $N$

## Acceleration

Occurs at endpoints $M$ and $N$


Special Cases of Starting Points of Motion
The equation,

$$
x=A \sin (\omega t+\theta)
$$

is a very general equation. Special cases can be obtained by using specific initial conditions.

Putting $\theta=0$ gives,

$$
x=A \sin \omega t
$$

(Motion starting at equilibrium position $x=0$ )
Putting $\theta=\pi / 2$ gives,

$$
\begin{array}{cc} 
& x=A \sin (\omega t+\pi / 2) \\
\Rightarrow & x=A \sin \omega t \cdot \cos \pi / 2+A \cos \omega t \cdot \sin \pi / 2 \\
\Rightarrow & x=A \sin \omega t \cdot 0+A \cos \omega t .1 \\
\Rightarrow & \begin{array}{c}
x=A \cos \omega t \\
\end{array} \\
\text { (Motion starting at right-hand endpoint } x=A)
\end{array}
$$

## A Useful Equation Connecting Velocity and Displacement

The defining equation for a particle executing SHM equation,

$$
a=-\omega^{2} x
$$

connects acceleration with displacement. A new equation can be obtained connecting velocity with displacement as follows.

$$
\begin{array}{ll} 
& \\
\therefore & v^{2}=\omega^{2} A^{2} \cos ^{2}(w t+\theta) \\
\Rightarrow & v^{2}=\omega^{2} A^{2}\left(1-\sin ^{2}(w t+\theta)\right) \\
\Rightarrow & \left.v^{2}=\omega^{2} A^{2}-\omega^{2} A^{2} \sin ^{2}(\omega t+\theta)\right) \\
\Rightarrow & v^{2}=\omega^{2} A^{2}-\omega^{2}(A \sin (w t+\theta))^{2} \\
\Rightarrow & v^{2}=\omega^{2} A^{2}-\omega^{2} x^{2} \\
\Rightarrow & v^{2}=\omega^{2}\left(A^{2}-x^{2}\right)
\end{array}
$$

## Example 1

Find the period of motion of a particle executing SHM according to the equation $a=-144 x$.

$$
\begin{array}{rlrl}
a & =-144 x \\
a & =-\omega^{2} x \\
& \therefore & \omega^{2} & =144 \\
\Rightarrow & & \frac{\omega}{}=12 \\
& & T & =\frac{2 \pi}{\omega} \\
& \therefore & T & =\frac{2 \pi}{12} \\
\Rightarrow & T & =\frac{\pi}{6} s
\end{array}
$$

## Example 2

Show that a particle whose displacement is governed by the equation $x=5 \sin 6 t$ executes SHM , and find the period of motion.

Also determine the magnitude of the maximum acceleration of the particle.

$$
\begin{array}{ll} 
& \\
& x=5 \sin 6 t \\
\therefore & \\
\Rightarrow & \ddot{x}=5(6) \cos 6 t \\
\Rightarrow & \ddot{x}=5(6)(-6) \sin 6 t \\
\Rightarrow & \ddot{x}=-36(5 \sin 6 t) \\
\Rightarrow & \ddot{x}=-36 x
\end{array}
$$

As $\ddot{x}=-\omega^{2} x(\omega=6)$, the particle executes SHM

$$
\begin{aligned}
& T=\frac{2 \pi}{\omega} \\
& \therefore \quad T=\frac{2 \pi}{6} \\
& \Rightarrow \quad \mathrm{~T}=\frac{\pi}{3} \mathrm{~s} \\
& \left|a_{\max }\right|=\omega^{2} A \\
& \therefore \quad\left|a_{\max }\right|=6^{2} .5 \\
& \Rightarrow \quad\left|a_{\text {max }}\right|=180 \mathrm{~m} \mathrm{~s}^{-2}
\end{aligned}
$$

## Example 3

A particle moves with SHM about centre of oscillation $O$ and a periodic time of $\pi / 8 \mathrm{~s}$.

Find the magnitude of the acceleration of the particle when it is two metres to the left of 0 .

$$
\begin{array}{rlrl} 
& & \omega & =\frac{2 \pi}{T} \\
& \therefore & \omega & =\frac{2 \pi}{\pi / 8} \\
\Rightarrow & & \frac{\omega}{}=16 \\
& & a & =-\omega^{2} x \\
& & |a| & =\left|\omega^{2} x\right|
\end{array}
$$

Taking right as positive, we have $x=-2 \mathrm{~m}$. So,

$$
\begin{array}{rlrl} 
& & |a| & =\left|16^{2}(-2)\right| \\
\Rightarrow & |a| & =512 \mathrm{~ms}^{-2}
\end{array}
$$

## Example 4

A SHO moves about a mean position $O$. The amplitude of the motion is 6 m and the period is $10 \pi \mathrm{~s}$.

Find the maximum speed of the particle and its velocities when 4 m from 0 .

$$
\left.\begin{array}{rlrl} 
& & \omega & =\frac{2 \pi}{T} \\
& \therefore & \omega & =\frac{2 \pi}{10 \pi} \\
\Rightarrow & & \omega & =\frac{1}{5} \\
& \therefore & & \left|v_{\text {max }}\right|
\end{array}\right)=\omega A
$$

Taking motion to be along the $x$-axis and $\underline{i}$ to be a unit vector in the positive $x$-direction,

$$
\underline{v}= \pm \frac{2}{\sqrt{5}} \mathrm{ims}^{-1}
$$

## Example 5

A particle moves with SHM about a mean position $O$. The amplitude of the motion is 85 cm and the time period is $\pi / 10 \mathrm{~s}$.

Find how far the particle is from $O$ when its speed is $8 \mathrm{~ms}^{-1}$.

$$
\begin{array}{ll} 
& \omega=\frac{2 \pi}{T} \\
\therefore & \omega=\frac{2 \pi}{\pi / 10} \\
\Rightarrow & \underline{\omega}=20 \\
& v^{2}=\omega^{2}\left(A^{2}-x^{2}\right) \\
\therefore & 8^{2}=20^{2}\left(0.85^{2}-x^{2}\right) \\
\Rightarrow & 0.85^{2}-x^{2}=64 / 400 \\
\Rightarrow & x^{2}=0.85^{2}-0.16 \\
\Rightarrow & x^{2}=0.5625 \\
\Rightarrow & x= \pm 0.75 \\
\therefore & \\
\therefore & \text { Distance from origin is } 0.75 \mathrm{~m}
\end{array}
$$

## Example 6

A particle moves with SHM about centre of oscillation O . When the particle is 180 cm from 0 , its speed is $9.6 \mathrm{~m} \mathrm{~s}^{-1} \mathrm{~m}$, and when it is 2.4 m from 0 , its speed is $7.2 \mathrm{~ms}^{-1}$.

Find the amplitude, angular frequency and period of the motion.

$$
v^{2}=\omega^{2}\left(A^{2}-x^{2}\right)
$$

Using $x=1.8 \mathrm{~m}, \mathrm{v}=9.6 \mathrm{~ms}^{-1}$ gives,

$$
9.6^{2}=\omega^{2}\left(A^{2}-1.8^{2}\right)
$$

Using $x=2.4 \mathrm{~m}, v=7.2 \mathrm{~ms}^{-1}$ gives,

$$
7.2^{2}=\omega^{2}\left(A^{2}-2.4^{2}\right)
$$

Solving the previous two equations for $\omega^{2}$ and equating the expressions obtained gives,

$$
\begin{aligned}
& \frac{9.6^{2}}{A^{2}-1.8^{2}}=\frac{7.2^{2}}{A^{2}-2.4^{2}} \\
& \therefore \quad 9.6^{2}\left(A^{2}-2.4^{2}\right)=7.2^{2}\left(A^{2}-1.8^{2}\right) \\
& \Rightarrow \quad 9.6^{2} A^{2}-530.8416=7.2^{2} A^{2}-167.9616 \\
& \Rightarrow \quad 40.32 A^{2}=362.88 \\
& \Rightarrow \quad A^{2}=9 \\
& \Rightarrow \quad A=3 \mathrm{~m} \\
& 9.6^{2}=\omega^{2}\left(A^{2}-1.8^{2}\right) \\
& \therefore \quad 9.6^{2}=\omega^{2}\left(3^{2}-1.8^{2}\right) \\
& \Rightarrow \quad 5.76 \omega^{2}=92.16 \\
& \Rightarrow \quad \omega^{2}=16 \\
& \Rightarrow \quad \omega=4 \mathrm{rads}^{-1} \\
& T=\frac{2 \pi}{\omega} \\
& \therefore \quad T=\frac{2 \pi}{4} \\
& \Rightarrow \quad T=\frac{\pi}{2} s
\end{aligned}
$$

## Example 7

A SHO of mass 50 g moves horizontally about a mean position 0 . When the oscillator is 10 cm from 0 , the horizontal force on the body is of magnitude 1.62 N .

Find the period of motion of the oscillator.
Let F be the magnitude of the force acting on the oscillator. Then,

$$
\begin{array}{rlrl} 
& & \underline{\mathbf{F}} & =m \underline{\underline{a}} \\
\Rightarrow & \underline{\mathbf{F}} & =-m \omega^{2} \times \underline{\mathbf{i}} \\
\therefore & & \mathbf{F} & =m \omega^{2} \times \\
\therefore & 1.62 & =(0.05)(0.1) \omega^{2} \\
\Rightarrow & 0.005 \omega^{2} & =1.62 \\
\Rightarrow & \omega^{2} & =324 \\
\Rightarrow & & \underline{\omega}=18 \\
& & T & =\frac{2 \pi}{\omega} \\
& & T & =\frac{2 \pi}{18} \\
& & T & =\frac{\pi}{9} s
\end{array}
$$

## Example 8

A body of mass 9 kg is placed on a rough horizontal surface with coefficient of static friction 3/7.

Determine whether or not the body will slide across the surface when the surface is moved horizontally with SHM of amplitude 60 cm and time period (a) $\pi \mathrm{s}$ (b) $2 \pi / 3 \mathrm{~s}$.

The body sliding across the surface means that the body moves relative to the surface in a given direction; this is equivalent to the surface moving relative to the body in the other direction. For definiteness, suppose the body is to move to the right relative to the surface (so the surface moves to the left). In order for the body to move right relative to the surface, the maximum value of the restoring force must be greater than the maximum static friction force (which acts to the left). The maximum restoring force occurs when the acceleration is a maximum and this occurs at an extremity of motion $(|x|=A)$. We require,

$$
\left.\begin{array}{rlrl} 
& & \left(F_{R E S}\right)_{\max } & >\left(F_{s}\right)_{\max } \\
& \therefore & m \omega^{2} A & >\mu_{s} m g \\
& \Rightarrow & & \omega^{2}
\end{array}\right)=\mu_{s} g / A .
$$

(a) $\mathrm{T}=\pi \Rightarrow \omega=2 \Rightarrow \omega^{2}=4$.

$$
\text { No sliding, as } \omega^{2} \ngtr 7 \Rightarrow\left(F_{R E s}\right)_{\text {max }} \ngtr\left(F_{s}\right)_{\text {max. }}
$$

(b) $T=2 \pi / 3 \Rightarrow \omega=3 \Rightarrow \omega^{2}=9$.

$$
\text { Sliding, as } \omega^{2}>7 \Rightarrow\left(F_{R E S}\right)_{\max }>\left(F_{s}\right)_{\max }
$$

## Example 9

A horizontal platform is made to move vertically up and down with SHM on a planet where the acceleration due to gravity is $12.1 \mathrm{~m} \mathrm{~s}^{-2}$.

If the period of motion is $2 \pi / 11 \mathrm{~s}$, show that any mass placed on the platform will leave the platform if the amplitude of motion is greater than 10 cm .

The motion is vertical, so we use the variable $y$ instead of $x$. A mass will leave the platform if the maximum value of the restoring force on the mass is greater than the weight of the mass. This will occur at an extremity of its motion ( $|y|=A$ ). Taking upwards to be positive, and denoting the magnitude of the maximum restoring force by $F$, the mass will leave the platform if,

$$
\begin{array}{ll} 
& F>W \\
\therefore & m \omega^{2} A>m g \\
\Rightarrow & A>g / \omega^{2} \\
\therefore & A>(121 / 10) /(2 \pi /(2 \pi / 11))^{2} \\
\Rightarrow & A>0.1 \mathrm{~m} \\
\Rightarrow & A>10 \mathrm{~cm}
\end{array}
$$

## Example 10

A particle moves with SHM about a mean position $O$. The particle is initially projected from $O$ with speed $3 \pi / 4 \mathrm{~m} \mathrm{~s}^{-1}$ and just reaches a point $Q, 6 \mathrm{~m}$ from 0 .

Find how far the particle is from $O$ six seconds after projection. Also determine the first three times after projection that the particle is a distance of three metres from $O$.

$$
\begin{array}{rlrl} 
& & v^{2} & =\omega^{2}\left(A^{2}-x^{2}\right) \\
\therefore & (3 \pi / 4)^{2} & =\omega^{2}\left(6^{2}-0^{2}\right) \\
\Rightarrow & \omega^{2} & =\left(9 \pi^{2} / 16\right) / 36 \\
\Rightarrow & \omega^{2} & =\pi^{2} / 64 \\
\Rightarrow & & \omega=\frac{\pi}{8} \\
& & x(t)=A \sin \omega t \\
& & x(t)=6 \sin (\pi t / 8)
\end{array}
$$

Six seconds after projection,

$$
\begin{aligned}
& x(6)=6 \sin (6 \pi / 8) \\
& \Rightarrow \quad x(6)=6 \sin (3 \pi / 4) \\
& \Rightarrow \quad x(6)=3 \sqrt{2} \mathrm{~m} \\
& x(t)=3 \\
& \therefore \quad 6 \sin (\pi \dagger / 8)=3 \\
& \Rightarrow \quad \sin (\pi \dagger / 8)=1 / 2 \\
& \therefore \quad \pi \dagger / 8=\pi / 6, \pi-\pi / 6, \pi+\pi / 6 \\
& \Rightarrow \quad \pi t / 8=\pi / 6,5 \pi / 6,7 \pi / 6 \\
& \Rightarrow \quad t=4 / 3 \mathrm{~s}, 20 / 3 \mathrm{~s}, 28 / 3 \mathrm{~s}
\end{aligned}
$$

## Blue Book

-pg. 439-441 Ex. 17 A Q 1 -17, 23, 29.

