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Force, Energy and Periodic Motion - Lesson 5

Gravitation

LI

- Know the vector and scalar forms of Newton's Law of Universal Gravitation (NLoUG).
- Know how Gravitational Field Strength is connected to NLoUG.
- Solve problems involving NLoUG.
- Solve problems involving Primaries and Satellites in circular orbits.

SC

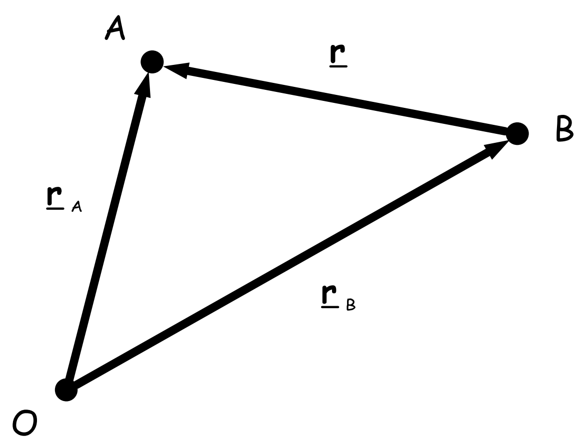
- Vectors.
- Centripetal motion.

Newton's (Inverse Square) Law of Universal Gravitation

> The force between 2 gravitating objects varies directly as the product of their masses and inversely as the square of the distance between their centres

'Inverse Square Law'

In more detail, start with a coordinate system with origin O and 2 gravitating objects A and B of masses m_A and m_B with respective position vectors $\underline{\mathbf{r}}_A$ and $\underline{\mathbf{r}}_B$ relative to O :



$$\underline{\mathbf{r}} = \underline{\mathbf{r}}_A - \underline{\mathbf{r}}_B$$

$$\underline{\mathbf{e}}_r = \frac{\underline{\mathbf{r}}}{|\underline{\mathbf{r}}|}$$

Unit vector from B to A

The gravitational force exerted on A due to B, denoted by $\underline{\mathbf{F}}_{AB}$, acts along the vector $-\underline{\mathbf{r}}$ from A to B and is given by,

$$\underline{\mathbf{F}}_{AB} = -\frac{G m_A m_B}{r^2}\,\underline{\mathbf{e}}_r$$

where $G \approx 6.67 \times 10^{-11}\ \mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$ is the Gravitational Constant (aka 'Big G').

The magnitude of this gravitational force is thus,

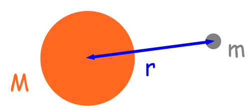
$$F_{AB} = \frac{G m_A m_B}{r^2}$$

In Newton's theory of gravitation, the gravitational force is not a direct contact force (it is a mysterious 'action at a distance' force). However, using the inverse-square law, it is easily seen that the force acting on A due to B has the same magnitude but opposite direction to the force acting on B due to A. In other words, Newton's 3rd Law applies to the inverse-square law :

$$\underline{\mathbf{F}}_{AB} = -\underline{\mathbf{F}}_{BA}$$

A connection between G and g can be obtained, where g is the acceleration experienced by an object due to the gravitational field of a gravitating body.

Consider a body of mass M exerting a gravitational force on a body of mass m, where the centres of these bodies are a distance r apart :



Taking the centre of the coordinate system to be at the centre of the body of mass M, the force on m is, by Newton's 2nd Law, $\underline{\mathbf{F}}_{mM} = m\,\underline{\mathbf{a}} = m\,\underline{\mathbf{g}} = -m\,g\,\underline{\mathbf{e}}_r$, where $\underline{\mathbf{e}}_r$ is a unit vector from M to m. This force also equals, by the inverse-square law,

$$\underline{\mathbf{F}}_{mM} = -\frac{G\,m\,M}{r^2}\,\underline{\mathbf{e}}_r$$

Hence,

$$\boxed{\underline{\mathbf{g}} = -\frac{G\,M}{r^2}\,\underline{\mathbf{e}}_r}$$

In magnitude form, this becomes,

$$g = \frac{G\,M}{r^2}$$

$$\Rightarrow \quad \boxed{g\,r^2 = G\,M = \text{constant}}$$

Thus, even though g and r are to be considered as variables (as r changes, g changes too), the product $g\,r^2$ is always constant (and equal to $G\,M$). We can thus write an alternative form for the above equation if G and M are not given or relevant :

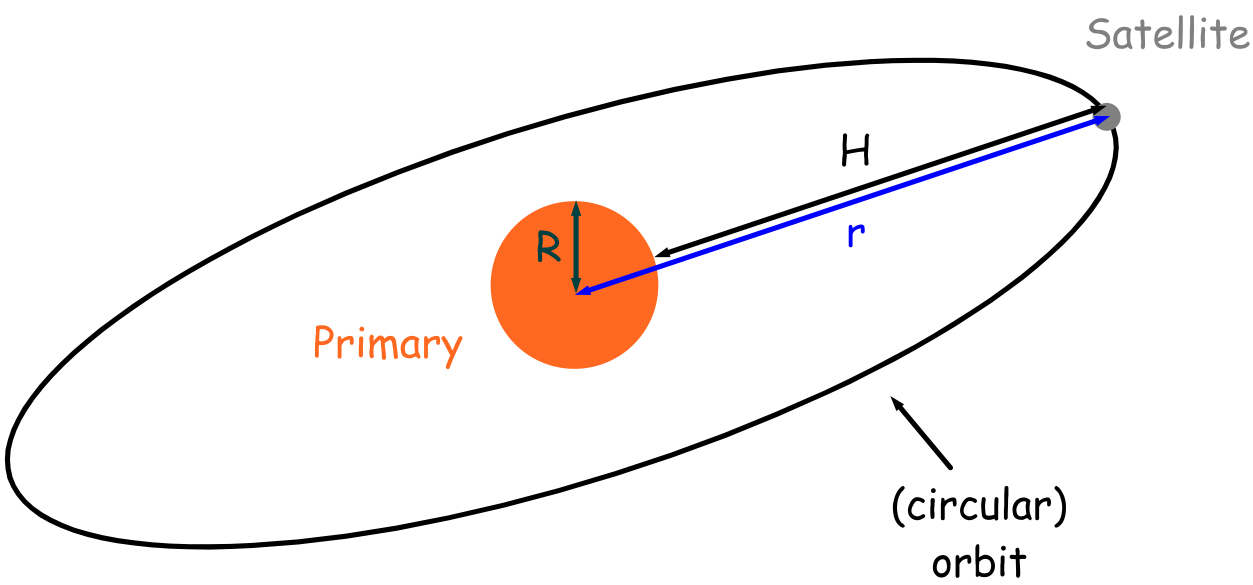
$$\boxed{g_1\,r_1^2 = g_2\,r_2^2}$$

In other topics (e.g. Projectile Motion), g is taken to be constant as it does not vary much close to the Earth's surface; but in this topic, g can vary considerably as r can vary considerably. It can also be very easily deduced from the relation $g\,r^2 = \text{constant}$ that as $r \longrightarrow \infty$, $g \longrightarrow 0$.

Jul 26-12:21

Satellite Problems

Problems involving objects orbiting other objects can be solved using the inverse-square law by making certain simplifying assumptions. The orbiting object ('satellite') is normally assumed to have a mass much smaller than the body ('primary') that it orbits. Also, the satellite is assumed to have a circular orbit and a constant speed.



Example 1

Find the magnitude of the gravitational force acting between two solid spheres of masses 200 g and 0 . 3 kg, with the distance between their centres being 150 cm.

$$F = \frac{G m_1 m_2}{r^2}$$

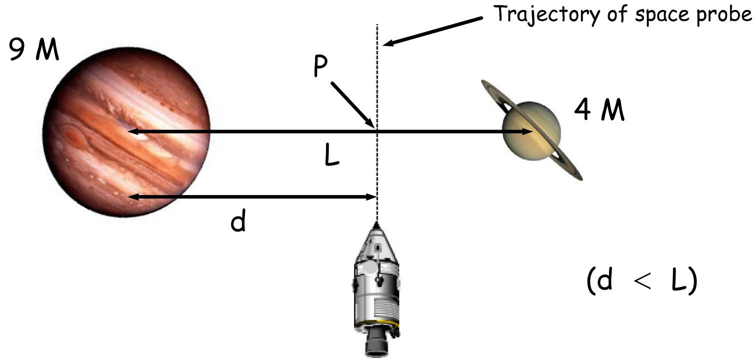
$$\therefore \quad F = \frac{(6.67 \times 10^{-11} \times 0.2 \times 0.3)}{(1.5)^2}$$

$$\Rightarrow \quad F = 1.778\,666\ldots \times 10^{-12}$$

$$\therefore \quad \boxed{F = 1.78 \times 10^{-12}\ \text{N}\ \ (2\ \text{s.f.})}$$

Example 2

The trajectory of a space probe passes between two planets of mass $9\,M$ and $4\,M$ as shown :



If the distance between the centres of the planets is L, determine the distance d (in terms of L) so that the space probe has zero net gravitational force from the two planets at P.

Taking the origin to be at P, the forces acting on the space probe (which has mass m, say) due to the larger and smaller planets respectively are, taking right as positive and $\underline{e}_r$ a unit vector along the positive horizontal,

$$\underline{F}_1 = \frac{G\,m\,(9\,M)}{d^2}\,\underline{e}_r \quad \text{and} \quad \underline{F}_2 = -\frac{G\,m\,(4\,M)}{(L-d)^2}\,\underline{e}_r$$

At P, $\underline{F}_1 + \underline{F}_2 = \underline{0}$ (net gravitational force equals zero). Hence,

$$\frac{9\,G\,M\,m}{d^2}\,\underline{e}_r - \frac{4\,G\,M\,m}{(L-d)^2}\,\underline{e}_r = \underline{0}$$

$$\therefore \qquad \frac{9}{d^2} = \frac{4}{(L-d)^2}$$

$$\Rightarrow \qquad 9\,(L-d)^2 = 4\,d^2$$

$$\Rightarrow \qquad 9\,L^2 - 18\,L\,d + 9\,d^2 = 4\,d^2$$

$$\Rightarrow \qquad 5\,d^2 - 18\,L\,d + 9\,L^2 = 0$$

$$\Rightarrow \qquad (5\,d - 3\,L)\,(d - 3\,L) = 0$$

$$\Rightarrow \qquad \underline{d = \frac{3\,L}{5}\,,\; d = 3\,L}$$

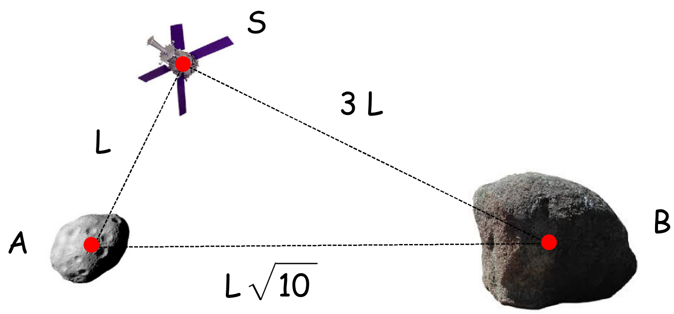
The second solution is not valid as $d < L$. Hence,

$$\boxed{d = \frac{3\,L}{5}}$$

Example 3

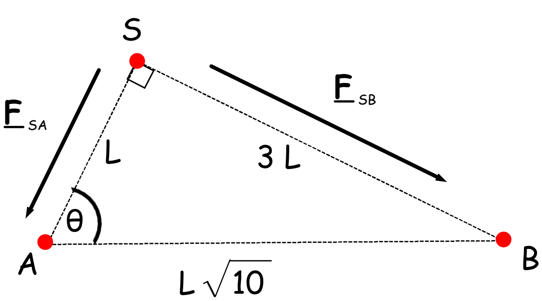
A space probe S experiences a gravitational attraction from 2 asteroids A and B, the mass of A being one-ninth the mass of B.

At one instant, the following situation exists :



Show that the magnitude of the x - component of the net force on S is half the magnitude of the y - component of the net force on S.

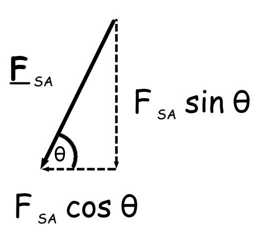
The Converse of Pythagoras' Theorem shows that triangle ASB is right-angled at S. We also have,



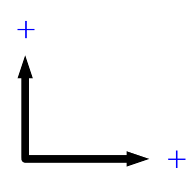
$$\sin\theta = \frac{3}{\sqrt{10}}$$

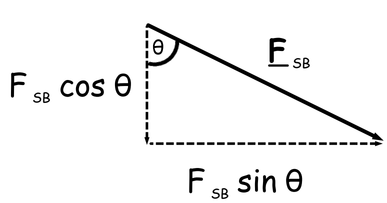
$$\cos\theta = \frac{1}{\sqrt{10}}$$

Resolving the forces acting on S into horizontal and vertical components gives,



$$\underline{F}_{SA} = (-F_{SA}\cos\theta)\,\underline{i} + (-F_{SA}\sin\theta)\,\underline{j}$$





$$\underline{F}_{SB} = (F_{SB}\sin\theta)\,\underline{i} + (-F_{SB}\cos\theta)\,\underline{j}$$

We also have,

$$F_{SA} = \frac{G\, m_A\, m_S}{L^2} \quad \text{and}$$

$$F_{SB} = \frac{G\, m_B\, m_S}{(3\,L)^2} = \frac{G\,(9\, m_A)\, m_S}{9\,L^2} = \frac{G\, m_A\, m_S}{L^2}$$

As $F_{SA} = F_{SB}$, we denote this common value by k.

The net force on S is,

$$\underline{\mathbf{F}}_{NET} = \underline{\mathbf{F}}_{SA} + \underline{\mathbf{F}}_{SB}$$

$$\therefore \quad \underline{\mathbf{F}}_{NET} = (-F_{SA}\cos\theta)\,\underline{\mathbf{i}} + (-F_{SA}\sin\theta)\,\underline{\mathbf{j}} + (F_{SB}\sin\theta)\,\underline{\mathbf{i}} + (-F_{SB}\cos\theta)\,\underline{\mathbf{j}}$$

$$\Rightarrow \quad \underline{\mathbf{F}}_{NET} = (F_{SB}\sin\theta - F_{SA}\cos\theta)\,\underline{\mathbf{i}} + (-F_{SA}\sin\theta - F_{SB}\cos\theta)\,\underline{\mathbf{j}}$$

$$\Rightarrow \quad \underline{\mathbf{F}}_{NET} = k\,(\sin\theta - \cos\theta)\,\underline{\mathbf{i}} + k\,(-\sin\theta - \cos\theta)\,\underline{\mathbf{j}}$$

$$\Rightarrow \quad \underline{\mathbf{F}}_{NET} = \frac{2\,k}{\sqrt{10}}\,\underline{\mathbf{i}} - \frac{4\,k}{\sqrt{10}}\,\underline{\mathbf{j}} = F_x\,\underline{\mathbf{i}} + F_y\,\underline{\mathbf{j}}$$

Hence,

$$F_x = \frac{2\,k}{\sqrt{10}} \Rightarrow |F_x| = \frac{2\,k}{\sqrt{10}}$$

$$F_y = -\frac{4\,k}{\sqrt{10}} \Rightarrow |F_y| = \frac{4\,k}{\sqrt{10}}$$

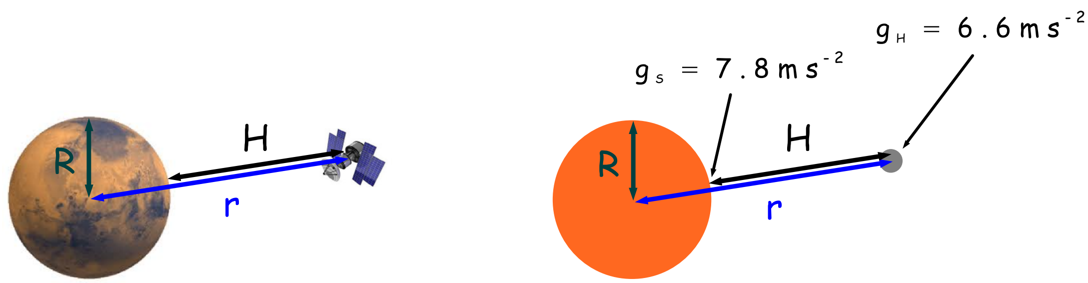
$$\therefore \quad \boxed{|F_x| = \frac{1}{2}\,|F_y|}$$

Jul 27-02:20

Example 4

A satellite orbits a planet (of radius 7 500 km) at a height H metres (at which height the acceleration due to gravity has magnitude $6.6\ m\ s^{-2}$) above the surface. The acceleration due to gravity at the planet's surface has magnitude $7.8\ m\ s^{-2}$.

Find H to the nearest metre.



(Diagram not to scale) Simplified diagram

We have 2 values for g and 1 value for distance. As G M is constant, we have,

$$g_H r^2 = g_S R^2$$

$$\therefore \quad 6.6\, r^2 = 7.8\,(7.5 \times 10^6)^2$$

$$\Rightarrow \quad r = \sqrt{\frac{7.8\,(7.5 \times 10^6)^2}{6.6}}$$

$$\Rightarrow \quad r = \sqrt{6.647\ldots \times 10^{13}}$$

$$\Rightarrow \quad \underline{r = 8\,153\,359.59\ldots}$$

$$H = r - R$$

$$\therefore \quad H = 8\,153\,359.59\ldots - 7\,500\,000$$

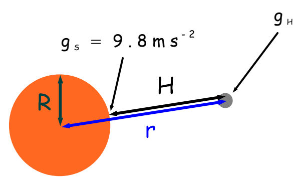
$$\Rightarrow \quad H = 653\,359.59\ldots$$

$$\therefore \quad \boxed{H = 653\,360 \text{ m (nearest metre)}}$$

Example 5

A satellite orbits the Earth (of radius 6 400 km) at a height of 920 km above the Earth's surface.

Calculate the magnitude of the acceleration due to gravity at this height and the satellite's orbital period (to the nearest minute).



We have 2 distances and 1 value for g. The distance of the satellite from the centre of the planet is the sum of H and R. So,

$$r = R + H$$

$$\therefore \quad r = 6\ 400\ 000 + 920\ 000$$

$$\Rightarrow \quad \underline{r = 7\ 320\ 000\ m}$$

$$g_H r^2 = g_s R^2$$

$$\therefore \quad g_H (7.32 \times 10^6)^2 = 9.8\,(6.4 \times 10^6)^2$$

$$\Rightarrow \quad g_H = \frac{9.8\,(6.4 \times 10^6)^2}{(7.32 \times 10^6)^2}$$

$$\Rightarrow \quad \underline{g_H = 7.491\ldots}$$

$$\therefore \quad \boxed{g_H = 7.5\ m\ s^{-2}\ (1\ d.p.)}$$

The angular speed is obtained through the central force,

$$m g_H = m r \omega^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{g_H}{r}}$$

$$\therefore \quad \omega = \sqrt{\frac{7.491\ldots}{7\ 320\ 000}}$$

$$\Rightarrow \quad \underline{\omega = 0.001\ 011\ 64\ldots}$$

The period is,

$$T = \frac{2\pi}{\omega}$$

$$\therefore \quad T = \frac{2\pi}{0.001\ 011\ 64\ldots}$$

$$\Rightarrow \quad T = 6\ 210.884\ 85\ldots\ s$$

$$\Rightarrow \quad T = 103.514\ 747\ldots\ min$$

$$\Rightarrow \quad T = 1.725\ 245\ 794\ldots\ h$$

$$\Rightarrow \quad T = 1\ h\ 43.5\ldots\ min$$

$$\therefore \quad \boxed{T = 1\ h\ 44\ min}$$

Sep 30-16:10

Example 6

A satellite orbits a primary of mass M. If the centre-centre distance between the satellite and primary is r and the orbital period of the satellite is T, show that $T^2 = (4\pi^2/G M)\, r^3$.

Equating the centripetal force to the gravitational force gives,

$$m r \omega^2 = \frac{G M m}{r^2}$$

$$\Rightarrow \quad m r^3 \omega^2 = G M m$$

$$\Rightarrow \quad r^3 (2\pi/T)^2 = G M$$

$$\Rightarrow \quad r^3 = (G M/4\pi^2)\, T^2$$

$$\Rightarrow \quad \boxed{T^2 = (4\pi^2/G M)\, r^3}$$

Gravitation Questions

1) Find the gravitational force acting betwen two electrons (mass of an electron $= 9.11 \times 10^{-31}$ kg) placed one metre apart.

2) The gravitational force between two masses (12 . 5 kg and 8 . 64 kg) is 8.004×10^{-10} N . Find the distance between their centres.

3) The gravitational force between two particles placed 620 cm apart is 5.336×10^{-10} N. If one particle has a mass of 4 kg, find the mass of the other particle.

4) Particle B of mass $6\sqrt{2}$ kg is placed 6 m horizontally to the right of particle A of mass 12 kg and 6 m vertically below particle S of mass 6 kg. Find the magnitude (as an exact value in terms of G) and direction of the net force on S due to A and B.

5) Mars has a mass of 6.42×10^{23} kg and a radius of 3.39×10^{6} m. Find the acceleration due to gravity at the surface of Mars.

6) Show that, for a satellite orbiting a primary of mass M at a distance r from the centre of the primary, $T^2 g = 4\pi^2 r$, where g is the acceleration due to gravity at distance r from the primary's centre. Hence find the period (to the nearest minute) of a satellite orbiting Mars at a height of 6 000 km above the Martian surface. Use data from the previous question as required.

Gravitation Answers

1) 5.54×10^{-71} N.

2) 3 m.

3) 76 . 88 kg.

4) $G\sqrt{5}$ N at 18° W of S.

5) 3.7 m s^{-2}.

6) 460 min or 7 h 40 min.