10 / 7 / 17

Force, Energy and Periodic Motion - Lesson 4

The Conical Pendulum

LI

- Derive the equations for a Conical Pendulum.
- Solve problems involving a Conical Pendulum.

SC

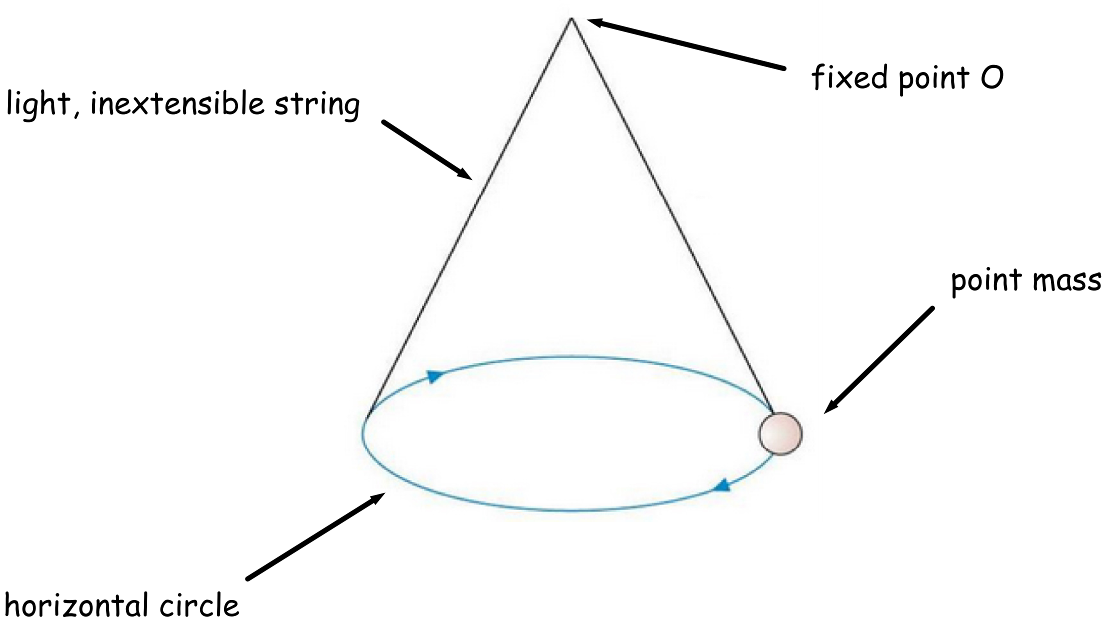
- Free-body diagrams (FBDs).
- Central force equation.

Jan 10-12:11

A Conical Pendulum is a point mass (moving under gravity), suspended by a light, inextensible string from a point O, where the mass moves in a horizontal circle with centre directly beneath O

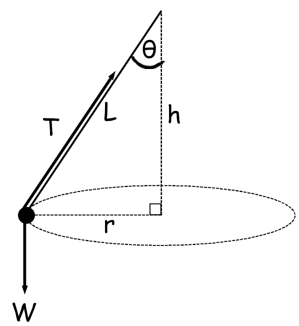
'Light' means negligible mass compared to the point mass; 'inextensible' means the string does not stretch.

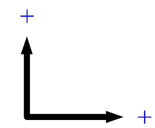
The term 'conical' arises from the fact that the horizontal circle and the string trace out a cone shape after one revolution

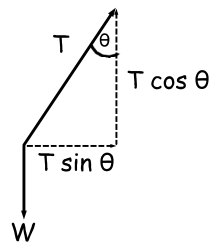


In a conical pendulum, there are always two forces that act on the point mass (aka bob) : string tension and gravity.

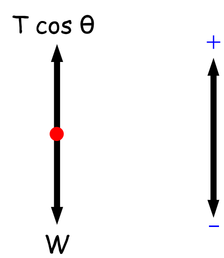
The horizontal component of the string tension provides the centripetal force that causes the bob to move in a horizontal circle.







<u>All Vertical Forces</u>



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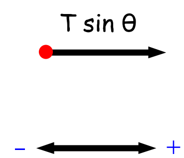

Net force $F_{NET} = F_c$ is to the right

Remembering that $W = m\,g$ and $F_c = \frac{m\,v^2}{r}$,

$$\boxed{\begin{aligned} T \sin \theta &= \frac{m\,v^2}{r} \\ T \cos \theta &= m\,g \end{aligned}}$$

The above two equations with some basic geometry and some equations for circular motion, are used to work out various quantities and relationships for a conical pendulum.

Many equations can be obtained connecting the quantities T, v, m, g, θ, r, h, L, ω, f (frequency) and P (period). Usually, 'T' is used for period, but we have here used it to denote tension.

The equations on the following two pages are not to be memorised; rather, they illustrate the types of equations that are obtained by analysing the FBD of a conical pendulum for any given question.

Tension equations

Using the first equation and $r = L \sin\theta$,

$$\frac{T r}{L} = \frac{m v^2}{r}$$

$$\Rightarrow \qquad \boxed{T = \frac{L m v^2}{r^2} = L m \omega^2}$$

where the relation $v = \omega r$ has been used.

Angle equations

Dividing the first equation by the second gives,

$$\boxed{\tan\theta = \frac{v^2}{g r} = \frac{r \omega^2}{g}}$$

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Period equations

Using $P = 2\pi/\omega$ and $v = \omega r$,

$$P = \frac{2\pi}{\omega}$$

$$\Rightarrow \qquad \boxed{P = \frac{2\pi r}{v}}$$

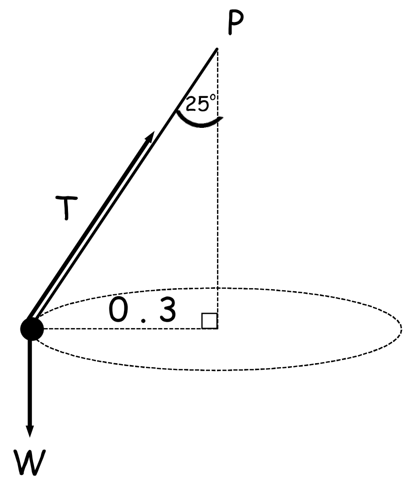
From the first angle equation $v^2 = g\, r \tan\theta$, this gives, upon simplification,

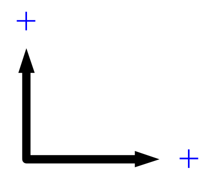
$$\boxed{P = 2\pi\sqrt{\frac{r}{g\tan\theta}}}$$

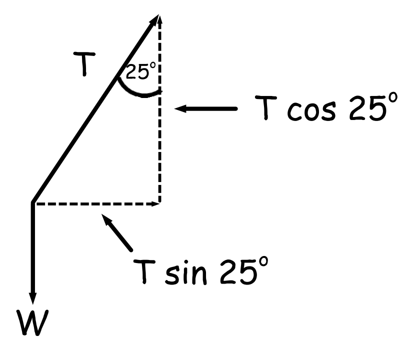
Example 1

A bob of mass 2 kg is attached to one end of a light, inextensible string, the other end being attached to a fixed point P.

If the bob moves in a horizontal circle of radius 30 cm with centre vertically below P, and the angle the string makes with the vertical from P is 25°, find the tension in the string, the angular velocity of the bob and the period of the bob.

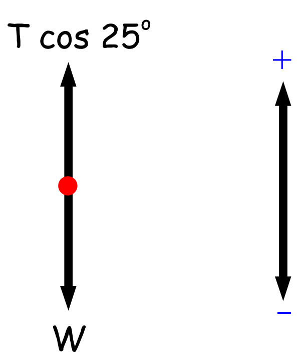


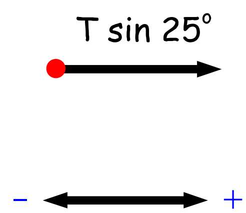




All Vertical Forces

All Horizontal Forces





Equating vertical components gives,

$$T \cos 25^{\circ} = m\,g$$

$$\Rightarrow \quad T = 2\,(9.8)/\cos 25^{\circ}$$

$$\Rightarrow \quad T = 21.62\,...$$

$$\therefore \quad \boxed{T = 21.6\text{ N } \ (1\text{ d.p.})}$$

The centripetal force is provided by the horizontal component of tension. So,

$$T \sin 25^{\circ} = m\,r\,\omega^{2}$$

$$\Rightarrow \quad \omega^{2} = (21.62\,...)\,(\sin 25^{\circ})/(2 \times 0.3)$$

$$\Rightarrow \quad \omega^{2} = 15.23\,...$$

$$\therefore \quad \boxed{\omega = 3.9\text{ rad s}^{-1} \ (1\text{ d.p.})}$$

The period is,

$$P = \frac{2\pi}{\omega}$$

$$\Rightarrow \quad P = \frac{2\pi}{3.9\,...}$$

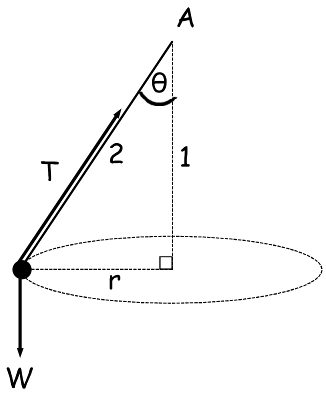
$$\Rightarrow \quad P = 1.60\,...$$

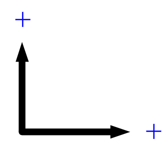
$$\therefore \quad \boxed{P = 1.6\text{ s } \ (1\text{ d.p.})}$$

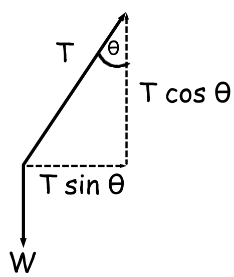
Example 2

A light, inextensible string of length 2 m is fixed at a point A and has the other end attached to an object of mass 3 kg. The mass rotates in a horizontal circle whose centre is one metre vertically below A.

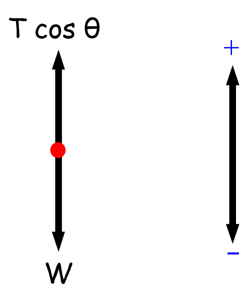
Show that the angular velocity of the mass is $\sqrt{g}$ and the string tension is 6 g.







All Vertical Forces



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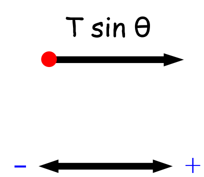
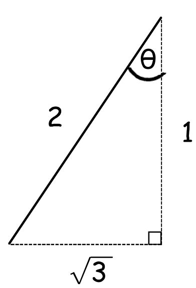


The right-angled triangle for length gives,



$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{1}{2}$$

Equating vertically gives,

$$T \cos \theta = m g$$

$$\therefore \quad T (1/2) = 3 g$$

$$\Rightarrow \quad \boxed{T = 6 g}$$

Net horizontal force is,

$$m r \omega^2 = T \sin \theta$$

$$\therefore \quad 3 \sqrt{3}\, \omega^2 = 6 g (\sqrt{3}/2)$$

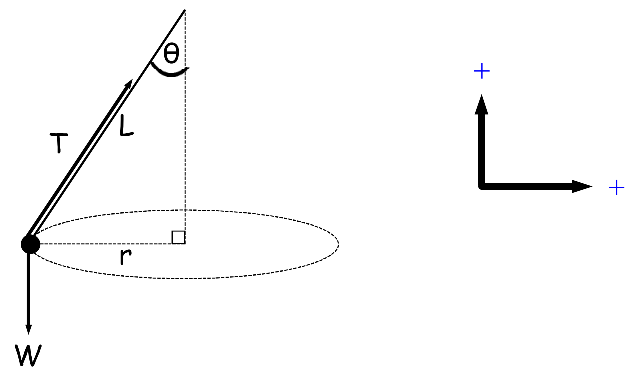
$$\Rightarrow \quad 3 \omega^2 = 3 g$$

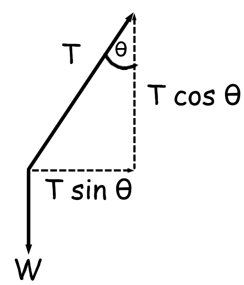
$$\Rightarrow \quad \boxed{\omega = \sqrt{g}}$$

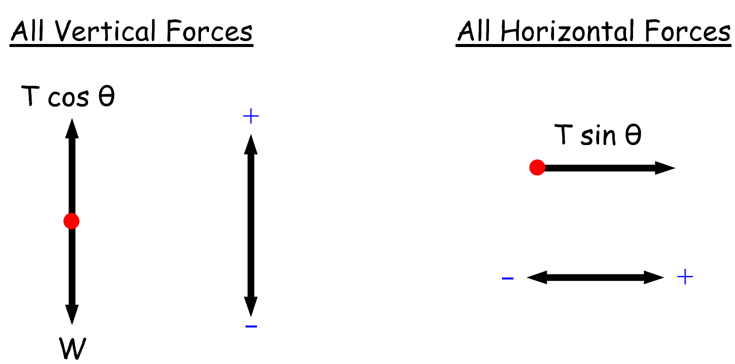
Example 3

A particle of mass M is attached to one end of a light, inextensible string of length L, the other end of which is fastened to a fixed point. The particle is made to describe circles of fixed radius r at a constant speed v.

Show that $g^2 r^4 = v^4 (L^2 - r^2)$.







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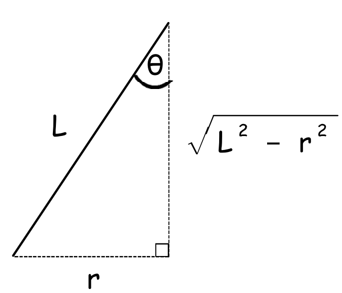


$$\sin \theta = \frac{r}{L}$$

$$\cos \theta = \frac{\sqrt{L^2 - r^2}}{L}$$

$$\tan \theta = \frac{r}{\sqrt{L^2 - r^2}}$$

We have,

$$T \sin\theta = \frac{m v^2}{r}$$

$$T \cos\theta = m g$$

$$\therefore \qquad \underline{\tan\theta = \frac{v^2}{g r}}$$

Equating this expression for $\tan\theta$ with the one obtained on the previous page gives,

$$\frac{r}{\sqrt{L^2 - r^2}} = \frac{v^2}{g r}$$

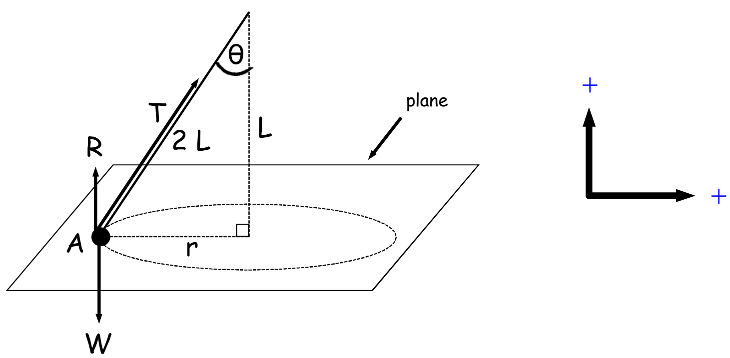
$$\Rightarrow \qquad g r^2 = v^2 \sqrt{L^2 - r^2}$$

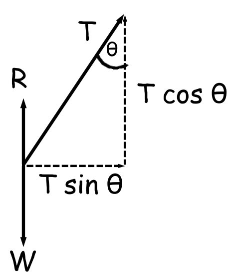
$$\Rightarrow \qquad \boxed{g^2 r^4 = v^4 (L^2 - r^2)}$$

Example 4

A particle of mass m, attached to the end A of a light, inextensible string describes a horizontal circle (with the string taut) on a smooth horizontal plane. The other end of the string is at a fixed height L above the centre of the horizontal circle.

If the angular velocity of the particle is ω, and the string has length $2L$, find the tension in the string and the reaction between the particle and the plane.





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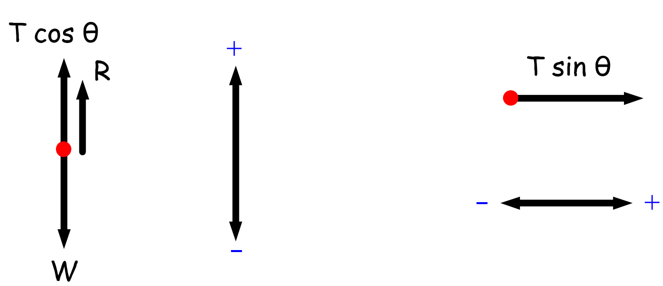

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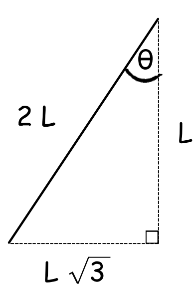


$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{1}{2}$$

Net horizontal force is,

$$T \sin\theta = m r \omega^2$$

$$\therefore \quad T(\sqrt{3}/2) = m L \sqrt{3}\, \omega^2$$

$$\Rightarrow \quad \boxed{T = 2 m L \omega^2}$$

Equating vertically gives,

$$R + T\cos\theta = m g$$

$$\therefore \quad R + 2 m L \omega^2 (1/2) = m g$$

$$\Rightarrow \quad \boxed{R = m g - m L \omega^2}$$

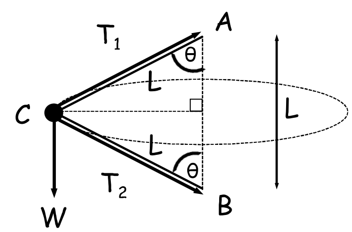
Example 5

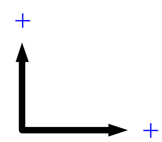
A light, inextensible string AB of length $2L$ has a particle attached to its midpoint C. The ends A and B of the string are fastened to two fixed points with A distance L vertically above B. With both parts of the string taut, the particle describes a horizontal circle about the line AB with constant angular speed ω.

If the tension in CA is four times that in CB, show that

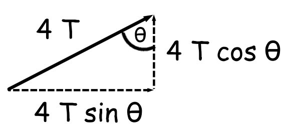
$$\omega = \sqrt{\frac{10g}{3L}}.$$

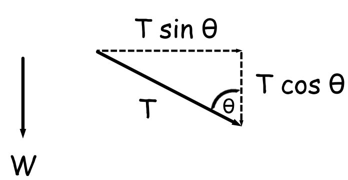
The particle is being made to move in a horizontal circle by effectively two strings CA and CB :





As the tension in T_1 is four times that in T_2, we let $T = T_2$. Then $T_1 = 4T$.





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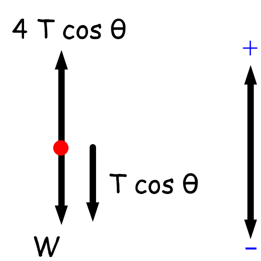

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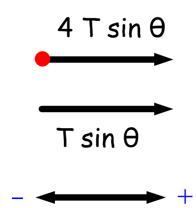

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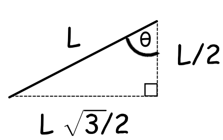

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\tan\theta = \sqrt{3}$$

Net horizontal force is,

$$4\,T\sin\theta + T\sin\theta = m\,r\,\omega^2$$

$$\Rightarrow \qquad \underline{5\,T\sin\theta = m\,r\,\omega^2}$$

Equating vertically gives,

$$4\,T\cos\theta = m\,g + T\cos\theta$$

$$\Rightarrow \qquad \underline{3\,T\cos\theta = m\,g}$$

Dividing these equations gives,

$$\frac{5\tan\theta}{3} = \frac{r\,\omega^2}{g}$$

$$\therefore \qquad \frac{5\sqrt{3}}{3} = \frac{L\sqrt{3}\,\omega^2}{2\,g}$$

$$\Rightarrow \qquad \boxed{\omega = \sqrt{\frac{10\,g}{3\,L}}}$$

Blue Book

- pg. 320-323 Ex. 13 C Q 1, 7, 8, 16, 17, 20, 23, 24.