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Unit 1 : Differential Calculus - Lesson 11

Related Rates

LI

- Work out related rates using a function.

SC

- Chain Rule.
- Implicit differentiation.

Related rates are rates of change (i.e. derivatives) that are connected by an equation

The connecting equation is differentiated and the Chain Rule is always used

If y is a function of u, and u is a function of x, then the Chain Rule is,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

If y is a function of u, u is a function of w, and w is a function of x, then the Chain Rule is,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dw} \times \frac{dw}{dx}$$

Example 1

A large spherical balloon is being deflated at a constant rate of $200\ cm^3\ s^{-1}$. If the balloon maintains its spherical shape during deflation, find how fast the radius is decreasing when the radius is 2 cm.

The volume V depends on the radius r and the radius changes with time t. So, the Chain Rule is,

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$V = (4/3)\pi r^3$$

$$\therefore \quad \frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\Rightarrow \quad -200 = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = -\frac{50}{\pi r^2}$$

$$\therefore \quad \left(\frac{dr}{dt}\right)_{r=2} = -\frac{50}{\pi (2)^2}$$

$$\Rightarrow \quad \boxed{\left(\frac{dr}{dt}\right)_{r=2} = -\frac{25}{2\pi}\ cm\ s^{-1}}$$

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Example 2

A particle is moving in a circle with centre the origin of the x-y plane and radius 5 m. When the particle is at the point (4, 3), the rate of change of the x - coordinate is 21 m s^{-1}.

Find the rate of change of the y - coordinate.

$$x^2 + y^2 = 25$$

Differentiating this equation implicitly gives,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\therefore \quad 2(4)(21) + 2(3)\frac{dy}{dt} = 0$$

$$\Rightarrow \quad 84 + 3\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \boxed{\frac{dy}{dt} = -28 \text{ m s}^{-1}}$$

AH Maths - MiA (2nd Edn.)

- pg. 196-9 Ex. 11.3
  Q 2, 6, 9, 10, 13.

Ex. 11.3

2 An oil spillage is spreading out on the garage floor.
It is circular in shape and its radius is expanding at the rate of 3 cms^{-1}.
How fast is the area growing when the radius is 50 cm?

6 A metal cube of edges x cm is heated. Each edge expands by 0.003 cmmin^{-1}.

a i Express the volume $V\ \text{cm}^3$ as a function of x.

ii Find an expression for the rate of change of volume with respect to edge length.

b Find the rate at which the volume is expanding with respect to time when the edge is 6 cm long.

9 According to Boyle's law the volume, $V\ \text{m}^3$, of a fixed mass of gas is inversely proportional to the pressure, $P\ \text{Nm}^{-2}$, of the gas.
In one particular case this leads to the relationship $PV = 500$.
The rate of change of the volume of the gas has been measured as 15 m^3s^{-1}.
Find the rate of change of pressure with respect to time when the volume of the gas is 50 m^3.

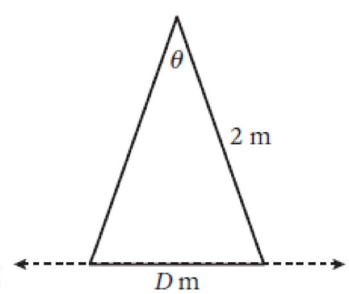
10 Climbing a mountain under ideal conditions, the temperature drops at the rate of 9.8 °C per 1000 m.

A climber models his ascent on the equation $h = 5t^2 + 150t$ where h is his height in metres and t is the time in hours.

a Find $\frac{dT}{dt}$ in degrees Celsius per minute.

b At what rate, in degrees Celsius per minute, does he experience the temperature dropping two hours into his ascent?

13 A set of stepladders is represented by an isosceles triangle whose equal sides are of length 2 m.
The feet of these lengths are D m apart, and held in place by a rope.
The rope snaps and the feet move apart. D increases by 20 cms^{-1}.
The apex angle is θ radians and its height at time t seconds is h metres.



a Express θ explicitly as a function of D.

b When the feet are 2.4 m apart, calculate the rate at which the angle is opening

i in radians per second ii in degrees per second.

c Calculate the rate at which the height, h metres, is decreasing when the feet are 3.3 m apart. Express your answer in cms^{-1}.

Answers to AH Maths (MiA), pg. 196-9, Ex. 11.3

2 300π cm^2/s

6 a i $V = (x + 0.003)^3$

ii $\frac{dV}{dx} = 3(x + 0.003)^2$

b 0·324 cm^3/min

9 -3 newton/m^2 per second.

10 a $\frac{\mathrm{d}T}{\mathrm{d}h} = -\left(\frac{9{\cdot}8}{1000}\right)$ °C/m $\Rightarrow$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\left(\frac{9{\cdot}8}{1000}\right)(10t + 150)\text{°C/hour}$$

$$= -\left(\frac{9{\cdot}8}{60\,000}\right)(10t + 150)\text{ °C/min}$$

[t measured in hours]

b $-\left(\frac{9{\cdot}8}{60\,000}\right)(20 + 150) = -0{\cdot}0278$ °C/min

13 a $\theta = 2\sin^{-1}\left(\frac{D}{4}\right)$

b i 0·125 radians per sec **ii** 7·16°/sec

c 14.6 cm/sec