6 / 7 / 17

*Force, Energy and Periodic Motion - Lesson 3*

# Central Forces and Banked Circular Motion

LI

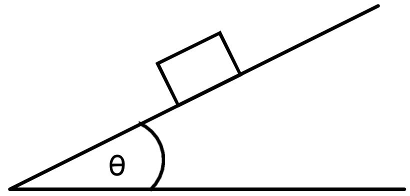
- Know and use the formula for centripetal force.
- Derive formulae for objects moving in a gravitational field round a banked/unbanked curve with/without friction.

SC

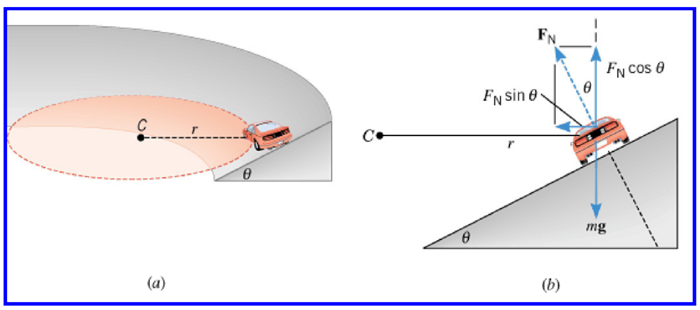
- Free-body diagrams (FBDs).

Jan 10-12:11

A banked curve is a path on a slope that is at a non-zero angle to the horizontal




Find conditions for object to move in a horizontal circle; this means no sliding (aka slipping or skidding).




There are 3 cases to consider for an object moving in a horizontal circle :

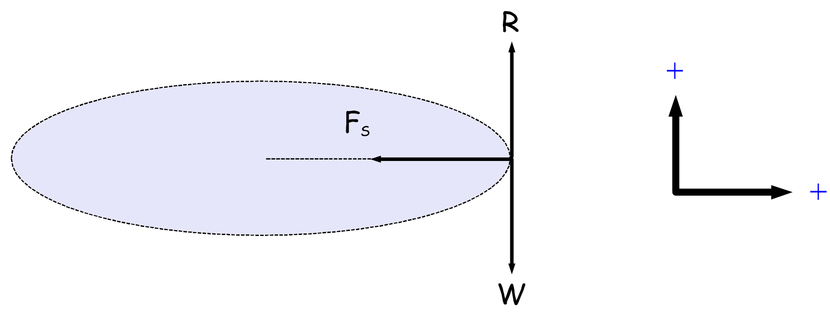
- unbanked, friction.
- banked, no friction.
- banked, friction.

The other possibility (unbanked, no friction) means that the object will gradually slide out and so will not travel in a circle.

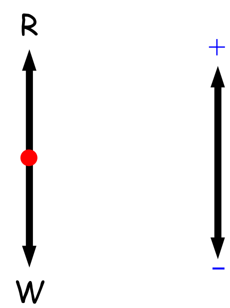
The friction referred to is always static friction; this may seem counterintuitive, as the object is moving (and so it seems it should be kinetic friction), but the motion is perpendicular to the radius of the circle. Static friction prevents the object from sliding up/down the slope;
if we were considering motion along the circular path that the object takes, then it would be kinetic friction.

For banked situations, it's best to resolve force vectors horizontally and vertically, as the centripetal force is horizontal and gravity is vertical.

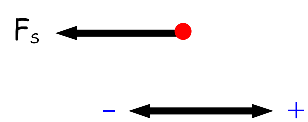
Rough, Unbanked Curve (no sliding out)




All Vertical Forces




All Horizontal Forces




Net force $F_{NET} = F_c$
is to the left

Equating vertically gives $R = W = m\,g$. As the object is to maintain a circular path, static friction is not overcome; so, we also have $F_s \leq \mu_s R$. Hence,

$$R = W = m\,g$$

$$F_s \leq \mu_s R$$

$$\therefore \quad \underline{F_s \leq \mu_s m\,g}$$

The net (centripetal) force is supplied by static friction,

$$F_c = F_s \leq \mu_s m\,g$$

$$\therefore \quad \frac{m\,v^2}{r} \leq \mu_s m\,g$$

$$\Rightarrow \quad \boxed{v \leq \sqrt{\mu_s\, g\, r}}$$

$$\therefore \quad \boxed{v_{max.} = \sqrt{\mu_s\, g\, r}}$$

The inequation tells us which values of $v$ are possible for the object to move in a circle. The biggest such value occurs when we have equality; for speeds greater than $v_{max.}$, the object will slide outwards (static friction will be overcome).

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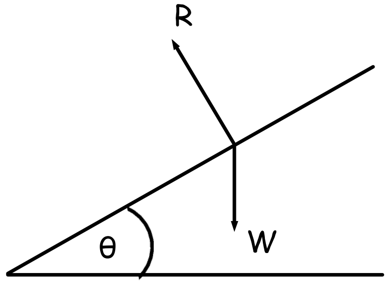
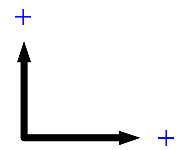
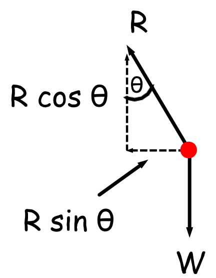
Smooth, Banked Curve (no sliding up and no sliding down)

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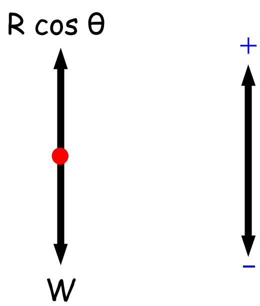


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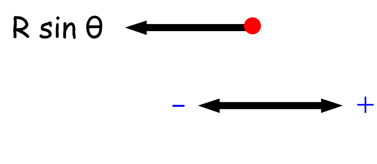



Net force $F_{NET} = F_c$ is to the left

The net vertical force is zero; the net horizontal force is the centripetal force. These give the two equations (respectively) :

$$R \cos \theta = W = m g$$

$$- R \sin \theta = - F_c = - \frac{m v^2}{r}$$

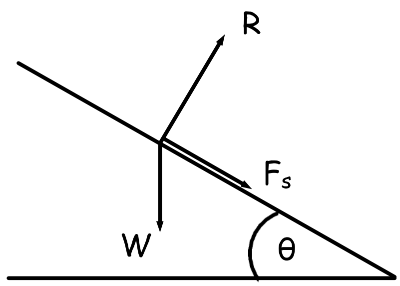
Dividing the second equation (after cancelling the negatives) by the first gives,

$$\frac{R \sin \theta}{R \cos \theta} = \frac{(m v^2/r)}{m g}$$

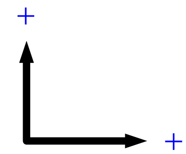
$$\therefore \qquad \tan \theta = \frac{v^2}{g r}$$

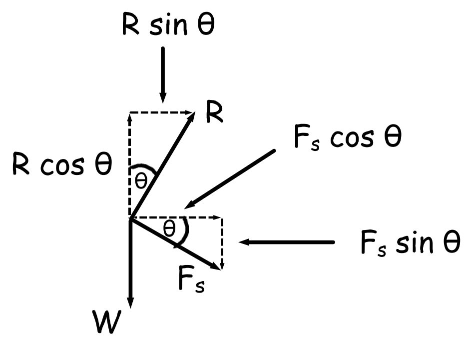
$$\Rightarrow \qquad \boxed{v = \sqrt{g r \tan \theta}}$$

## Rough, Banked Curve (just prevented from sliding up)

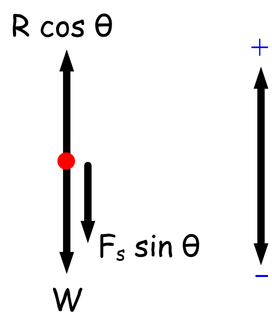



'just prevented from sliding up' means friction is down the slope




All Vertical Forces

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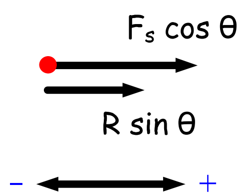



Net force $F_{NET} = F_C$ is to the right

The net vertical force is zero. So,

$$W + F_s \sin\theta = R\cos\theta$$

$$\Rightarrow \quad m\,g + F_s \sin\theta = R\cos\theta$$

$$\Rightarrow \quad R\cos\theta - m\,g = F_s \sin\theta$$

$$\therefore \quad R\cos\theta - m\,g \leq \mu_s R\sin\theta$$

$$\Rightarrow \quad m\,g \geq R\cos\theta - \mu_s R\sin\theta$$

$$\Rightarrow \quad \underline{m \geq (R/g)(\cos\theta - \mu_s \sin\theta)}$$

The net horizontal force is the centripetal force, provided by $F_s \cos\theta$ and $R \sin\theta$. So,

$$\frac{m v^2}{r} = R\sin\theta + F_s\cos\theta$$

$$\Rightarrow \quad \frac{m v^2}{r} - R\sin\theta = F_s\cos\theta$$

$$\therefore \quad \frac{m v^2}{r} - R\sin\theta \leq \mu_s R\cos\theta$$

$$\Rightarrow \quad \frac{m v^2}{r} \leq R\sin\theta + \mu_s R\cos\theta$$

$$\Rightarrow \quad \underline{m \leq (R\,r/v^2)(\sin\theta + \mu_s\cos\theta)}$$

Combining the two inequations for $m$ gives,

$$(R/g)(\cos\theta - \mu_s\sin\theta) \leq m \leq (R\,r/v^2)(\sin\theta + \mu_s\cos\theta)$$

$$\therefore \quad (R/g)(\cos\theta - \mu_s\sin\theta) \leq (R\,r/v^2)(\sin\theta + \mu_s\cos\theta)$$

$$\Rightarrow \quad v^2(\cos\theta - \mu_s\sin\theta) \leq g\,r(\sin\theta + \mu_s\cos\theta)$$

$$\Rightarrow \quad \boxed{v \leq \sqrt{\frac{g\,r(\sin\theta + \mu_s\cos\theta)}{(\cos\theta - \mu_s\sin\theta)}}}$$

$$\therefore \quad \boxed{v_{max.} = \sqrt{\frac{g\,r(\sin\theta + \mu_s\cos\theta)}{(\cos\theta - \mu_s\sin\theta)}}}$$

The inequation tells us which values of $v$ are possible for the object to move in a circle. The biggest such value occurs when we have equality; for speeds greater than $v_{max.}$, the object will slide upwards (static friction will be overcome).

Also note that, when $\mu_s = 0$, we have $v \leq \sqrt{g\,r\tan\theta}$ .

Rough, Banked Curve (just prevented from sliding down)

In a similar way, it can be shown that the speed required to just prevent the object from sliding down is,

$$\boxed{v \geq \sqrt{\frac{g\, r\, (\sin\theta - \mu_s \cos\theta)}{(\cos\theta + \mu_s \sin\theta)}}}$$

The minimum such speed will be,

$$\boxed{v_{min.} = \sqrt{\frac{g\, r\, (\sin\theta - \mu_s \cos\theta)}{(\cos\theta + \mu_s \sin\theta)}}}$$

Note that, for both situations (preventing sliding up and preventing sliding down), when $\mu_s = 0$, we have $v \leq \sqrt{g\, r \tan\theta}$ and $v \geq \sqrt{g\, r \tan\theta}$, respectively. Hence, to move in a circle, we must have $v = \sqrt{g\, r \tan\theta}$ , which reduces to a previous case (Smooth, Banked Curve (no sliding down or sliding up)).

Also, when $\theta = 0$, only the first of the cases applies ('sliding downwards' becomes 'sliding inwards', but this can't happen when $\theta = 0$). This then reduces to a previous case (Rough, Unbanked Curve (no sliding out)).

The results on the previous 5 pages need not be memorised; rather, they illustrate general techniques for solving specific problems.

For any given question, decide if it's a friction/no friction problem. Then decide if it's a banked/unbanked problem.

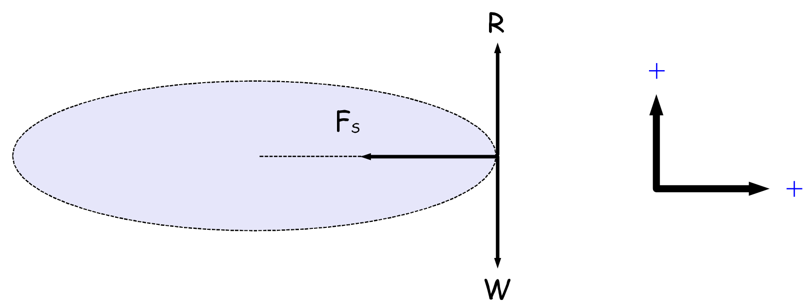
It is essential to draw a FBD, once positive directions have been chosen.

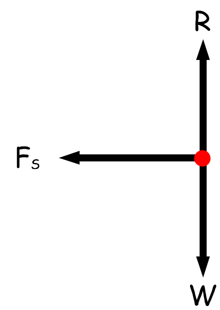
There are always 2 forces present, namely, weight (W) and normal reaction (N or R). Then there may be friction.

Vertical forces are balanced, and there is a net horizontal force (centripetal force).

Example 1

A vehicle travels on a rough road describing a horizontal circle. If the radius of the track is 100 m and there is no tendency for the vehicle to slip outwards, find the maximum speed that the vehicle can negotiate the curve if the coefficient of static friction is 0 . 3.




Equating vertically gives $R = W = m\,g$. The maximum speed will occur when the object is on the point of slipping outwards. Then we have, $(F_s)_{max.} = \mu_s R$. Hence,

$$R = W = m\,g$$

$$(F_s)_{max.} = \mu_s R$$

$$\therefore \quad \underline{(F_s)_{max.} = \mu_s m\,g}$$

The net (centripetal) force is supplied by static friction,

$$F_c = (F_s)_{max.} = \mu_s m\,g$$

$$\therefore \quad \frac{m\,v^2}{r} = \mu_s m\,g$$

$$\Rightarrow \quad v = \sqrt{\mu_s g\, r}$$

$$\therefore \quad v = \sqrt{(0\,.\,3)\,(9\,.\,8)\,(100)}$$

$$\Rightarrow \quad v = 17\,.\,14\,.\,.\,.$$

$$\therefore \quad \boxed{v = 17\,.\,1 \text{ m s}^{-1} \text{ (1 d.p.)}}$$

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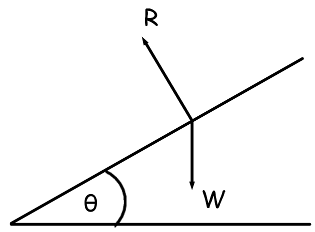
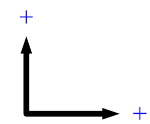
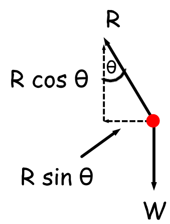
Example 2

A car travels round a circular bend of radius 500 m on a smooth road which is banked at an angle θ to the horizontal.

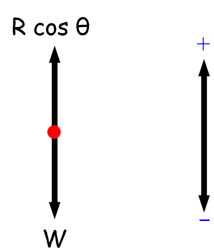
If the car has no tendency to slip when travelling at a speed of 45 m s$^{-1}$, find θ.



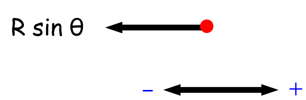




All Vertical Forces




All Horizontal Forces




Net force $F_{NET} = F_c$
is to the left

The net vertical force is zero; the net horizontal force is the centripetal force. These give the two equations (respectively) :

$$R \cos\theta = W = m g$$

$$-R \sin\theta = -F_c = -\frac{m v^2}{r}$$

Dividing the second equation (after cancelling the negatives) by the first gives,

$$\frac{R \sin\theta}{R \cos\theta} = \frac{(m v^2/r)}{m g}$$

$$\Rightarrow \quad \tan\theta = \frac{v^2}{g r}$$

$$\therefore \quad \tan\theta = \frac{45^2}{(9.8)(500)}$$

$$\Rightarrow \quad \tan\theta = 0.413\ldots$$

$$\Rightarrow \quad \boxed{\theta = 22.5^\circ \text{ (1 d.p.)}}$$

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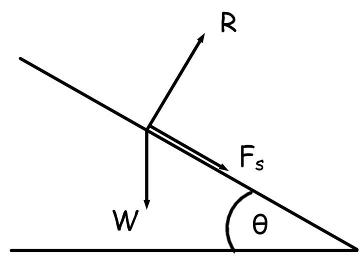
Example 3

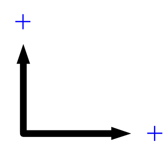
A car travels around a bend in a road which is a circular arc of radius 62 . 5 m. The road is banked at an angle of $\tan^{-1}(5/12)$ to the horizontal.

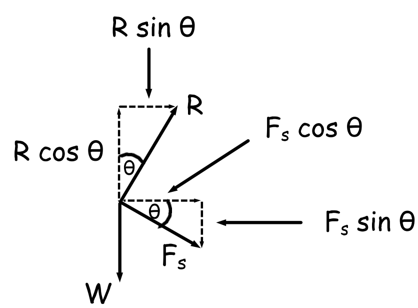
If the coefficient of static friction between the tyres of the car and the road surface is 0 . 4, find the greatest and least speeds at which the car can be driven around the bend without any slipping occuring.

Greatest speed :

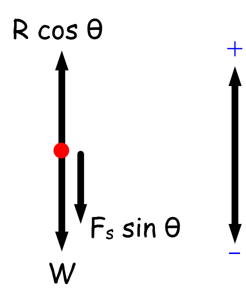
The maximum speed required will occur when friction acts down the slope.



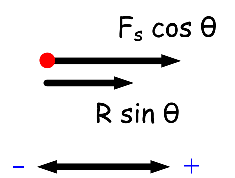




All Vertical Forces




All Horizontal Forces




Net force $F_{NET} = F_c$ is to the right

These give the equations,

$$W + F_s \sin\theta = R\cos\theta$$

$$\frac{m v^2}{r} = R\sin\theta + F_s\cos\theta$$

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To just prevent slipping up (maximum speed), $F_s = \mu_s R$. So,

$$m g + \mu_s R \sin\theta = R \cos\theta$$

$$\frac{m v^2}{r} = R \sin\theta + \mu_s R \cos\theta$$

Solving each of the above equations for $R$ gives, respectively,

$$R = \frac{m g}{\cos\theta - \mu_s \sin\theta}$$

$$R = \frac{m v^2}{r(\sin\theta + \mu_s \cos\theta)}$$

Equating these expressions and solving for $v$ gives,

$$v = \sqrt{\frac{g r (\sin\theta + \mu_s \cos\theta)}{(\cos\theta - \mu_s \sin\theta)}}$$

$$\Rightarrow \quad v = \sqrt{\frac{g r (\tan\theta + \mu_s)}{(1 - \mu_s \tan\theta)}}$$

$$\therefore \quad v = \sqrt{\frac{(9.8)(62.5)(5/12 + 0.4)}{(1 - (0.4)(5/12))}}$$

$$\Rightarrow \quad \underline{v = 24.5 \text{ m s}^{-1}}$$

<u>Least speed :</u>

The minimum speed required will occur when friction acts up the slope. This small difference gives,

$$v = \sqrt{\frac{g r (\sin\theta - \mu_s \cos\theta)}{(\cos\theta + \mu_s \sin\theta)}}$$

$$\Rightarrow \quad v = \sqrt{\frac{g r (\tan\theta - \mu_s)}{(1 + \mu_s \tan\theta)}}$$

$$\therefore \quad v = \sqrt{\frac{(9.8)(62.5)(5/12 - 0.4)}{(1 + (0.4)(5/12))}}$$

$$\Rightarrow \quad \underline{v = 2.96 \text{ m s}^{-1} \text{ (2 d.p.)}}$$

Greatest speed is $25 \text{ m s}^{-1}$; least speed is $2.96 \text{ m s}^{-1}$ (2 d.p.)

Jul 5-11:17

Blue Book

- pg. 320-323 Ex. 13 C Q 2-6, 11-15, 18, 19.