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Unit 1 : Differential Calculus - Lesson 10

Planar Motion

LI

- Work out kinematical quantities.

SC

- Vectors.
- Parametric differentiation.

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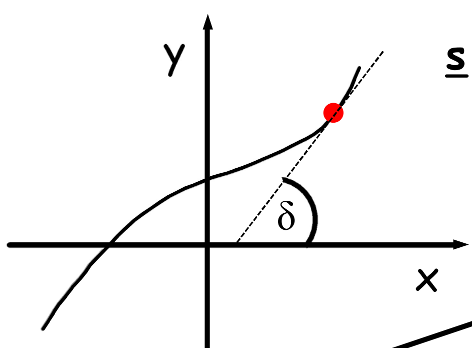
Planar motion means motion in a (2D) plane (usually the x-y plane)

The displacement (denoted variously by $\underline{\mathbf{s}}$ or $\underline{\mathbf{r}}$) of a particle is its position vector from the origin of a coordinate system :



$$\underline{\mathbf{s}}(t) = x(t)\,\underline{\mathbf{i}} + y(t)\,\underline{\mathbf{j}} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\tan\delta = \frac{y(t)}{x(t)}$$

Direction that $\underline{\mathbf{s}}(t)$ makes with the positive x - axis

Distance is the magnitude of displacement :

$$s(t) = |\underline{\mathbf{s}}(t)| = \sqrt{x^2 + y^2}$$

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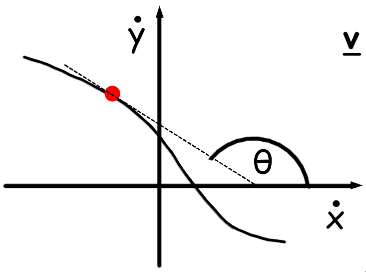
The velocity (denoted by $\underline{v}$) of a particle is the time-derivative of the particle's displacement :

$$\underline{v} = \frac{d\underline{s}}{dt} = \dot{\underline{s}}$$



$$\underline{v}(t) = \dot{x}(t)\,\underline{i} + \dot{y}(t)\,\underline{j} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

$$\tan\theta = \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{dy}{dx}$$

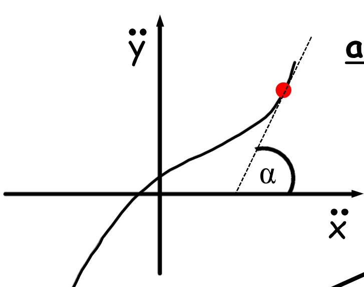
Direction that $\underline{v}(t)$ makes with the positive x - axis (aka direction of motion)

Speed is the magnitude of velocity :

$$v(t) = |\underline{v}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$$

The acceleration (denoted by $\underline{a}$) of a particle is the time-derivative of the particle's velocity :

$$\underline{a} = \frac{d\underline{v}}{dt} = \dot{\underline{v}}$$



$$\underline{a}(t) = \ddot{x}(t)\,\underline{i} + \ddot{y}(t)\,\underline{j} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix}$$

$$\tan\alpha = \frac{\ddot{y}(t)}{\ddot{x}(t)}$$

Direction that $\underline{a}(t)$ makes with the positive x - axis

Magnitude of acceleration :

$$a(t) = |\underline{a}(t)| = \sqrt{\ddot{x}^2 + \ddot{y}^2}$$

Unless otherwise stated, the following units are assumed :

- Displacement - metres (m).
- Velocity - metres per second ($m\ s^{-1}$).
- Acceleration - metres per second squared ($m\ s^{-2}$).

$$\underline{s}(t) \xrightarrow{\frac{d}{dt}} \underline{v}(t) \xrightarrow{\frac{d}{dt}} \underline{a}(t)$$

Useful Terminology

Constant - unchanging

At rest - velocity $= 0$

Initially - time starts at 0

At origin - displacement $= 0$

Unless otherwise stated, a particle will be assumed to commence its motion from the origin at $t = 0$

Example 1

The equations of motion of a particle moving in a plane are,

$$x(t) = t^3 - 3t \ , \quad y(t) = t^2 + t$$

Find the displacement of the particle after 2 seconds and also the magnitude and direction of its velocity after 3 seconds.

$$\underline{s}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\therefore \quad \underline{s}(t) = \begin{pmatrix} t^3 - 3t \\ t^2 + t \end{pmatrix}$$

$$\therefore \quad \underline{s}(2) = \begin{pmatrix} 2^3 - 3(2) \\ 2^2 + 2 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\underline{s}(2) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}}$$

$$x(t) = t^3 - 3t \ , \quad y(t) = t^2 + t$$

$$\therefore \quad \dot{x}(t) = 3t^2 - 3 \ , \quad \dot{y}(t) = 2t + 1$$

$$\underline{\mathbf{v}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

$$\therefore \quad \underline{\mathbf{v}}(t) = \begin{pmatrix} 3t^2 - 3 \\ 2t + 1 \end{pmatrix}$$

$$\therefore \quad \underline{\mathbf{v}}(3) = \begin{pmatrix} 3(3)^2 - 3 \\ 2(3) + 1 \end{pmatrix}$$

$$\Rightarrow \quad \underline{\underline{\mathbf{v}}(3) = \begin{pmatrix} 24 \\ 7 \end{pmatrix}}$$

$$\therefore \quad v(3) = \sqrt{24^2 + 7^2}$$

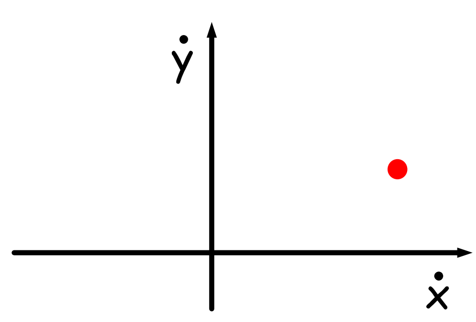
$$\Rightarrow \quad \boxed{v(3) = 25\ \mathrm{m\,s^{-1}}}$$

$$\tan\theta = \frac{\dot{y}(3)}{\dot{x}(3)}$$

$$\therefore \quad \tan\theta = 7/24$$

$$\Rightarrow \quad \underline{RA = 16.3^\circ}$$

$$\therefore \quad \boxed{\theta = 16.3^\circ}$$



Example 2

The equations of motion of a particle moving in a plane are,

$$x(t) = \sin 2t \;,\quad y(t) = \cos 2t$$

Show that the magnitude of the acceleration of the particle is constant.

$$x(t) = \sin 2t \;,\quad y(t) = \cos 2t$$

$$\therefore \quad \dot{x}(t) = 2\cos 2t \;,\quad \dot{y}(t) = -2\sin 2t$$

$$\Rightarrow \quad \ddot{x}(t) = -4\sin 2t \;,\quad \ddot{y}(t) = -4\cos 2t$$

$$\underline{a}(t) = \ddot{x}(t)\,\underline{i} + \ddot{y}(t)\,\underline{j}$$

$$\therefore \quad \underline{a}(t) = (-4\sin 2t)\,\underline{i} + (-4\cos 2t)\,\underline{j}$$

$$\therefore \quad |\underline{a}(t)| = \sqrt{(-4\sin 2t)^2 + (-4\cos 2t)^2}$$

$$\Rightarrow \quad |\underline{a}(t)| = \sqrt{16\sin^2 2t + 16\cos^2 2t}$$

$$\Rightarrow \quad |\underline{a}(t)| = \sqrt{16(\sin^2 2t + \cos^2 2t)}$$

$$\Rightarrow \quad |\underline{a}(t)| = \sqrt{16(1)}$$

$$\Rightarrow \quad \boxed{|\underline{a}(t)| = 4\ \text{m s}^{-2} \Rightarrow |\underline{a}(t)| \text{ is constant}}$$

AH Maths - MiA (2nd Edn.)

- pg. 193-4 Ex. 11.2
  Q 1, 3, 6, 8, 9, 10 a, b.

Ex. 11.2

- t represents time in seconds; x and y are distances in metres.
- Work in radian measure when dealing with trigonometric functions.
- Work to 3 significant figures where appropriate.

1 A particle moves in a plane according to the parametric equations $x = \frac{t^3}{\pi} + \sin t, y = t^2 - \cos t$.

a Find the speed of the particle, and the direction of motion, when $t = \pi$.

b Find the magnitude of the acceleration at the same time.

3 A particle moves in a plane according to the equations
$x = \ln(t + 2) - 0.69, y = \ln(t + 3) + 1$

a i Use the chain rule to establish an expression in t for $\frac{dy}{dx}$.

ii Hence show that when $t > 0, \frac{dy}{dx} > 0$, and so the displacement is always increasing.

b How far did the particle travel between the first and fourth seconds?

c Find the magnitude and direction of the velocity at the first second.

d Find the magnitude and direction of the acceleration initially.

6 The London Eye is a tourist attraction on the banks of the Thames.
It is a ferris wheel that travels slow enough to allow people to step on and off without it stopping. The movement of a point on the circumference can be modelled by
$x = 65\cos\left(\frac{\pi t}{15} - \frac{\pi}{2}\right), y = 65\sin\left(\frac{\pi t}{15} - \frac{\pi}{2}\right) + 70$
where t is the number of minutes since the point was last at the bottom of the wheel.

a How high above the ground is the bottom of the wheel?
[That is, what is y when $x = 0$?]

b How long does it take to make a complete revolution?

c i Show that its speed is a constant (that is, independent of t) and give its value.

ii Describe its velocity after 10 min.

d Describe its acceleration after 15 min.

8 The velocity of a particle can be modelled by $\dot{x} = \frac{e^t - e^{-t}}{2}, \dot{y} = \frac{e^t + e^{-t}}{2}$.

a i Find parametric equations to give the position at time t.

ii Find the position of the particle initially if the constants generated by integration are both zero.

b Express, in terms of e^{2t},

i the speed

ii the tangent of the angle that the direction of motion makes with the x-direction.

c Find the acceleration when $t = 5$.

9 On 14th February, a laser light was made to play on a wall tracing out a design by following the equations of motion $x = \cos^3 t, y = \sin t - \sin^2 t$ where t is the time in tenths of a second and x and y are measured in metres.

a What is the position of the starting point?

b Calculate the velocity when $t = 10$.

10 A particle's motion is described by $x = 1 + \cos t - \sin t, y = 1 - \cos t + \sin t$.

a Show that at all times t, the direction of the velocity and that of the acceleration are the same.

b Show that $|v(t)|^2 + |a(t)|^2 = 4 \; \forall \; t$.

Answers to AH Maths (MiA), pg. 193-4, Ex. 11.2

1 a Speed 10·5 m/s direction 0·641 radians to x-direction

b $\sqrt{37}$

3 a i $\dfrac{dy}{dx} = \dfrac{t+2}{t+3}$ **ii** $t > 0 \Rightarrow \dfrac{dy}{dx} > 0$

b 0·724 m

c $|v(1)| = 0{\cdot}417\ \text{ms}^{-1}$; 0.644 radians

d $|a(0)| = 0{\cdot}274\ \text{ms}^{-1}$; 3.560 radians

6 a 5 m **b** 30 min

c i $|v| = \sqrt{\left(\dfrac{65\pi}{15}\right)^2\left[\sin^2\left(\dfrac{\pi t}{15} - \dfrac{\pi}{2}\right) + \cos^2\left(\dfrac{\pi t}{15} - \dfrac{\pi}{2}\right)\right]}$

$= \dfrac{65\pi}{15} \approx 13{\cdot}6$ m/min

ii $v(10) = \begin{pmatrix} -6{\cdot}81 \\ 11{\cdot}8 \end{pmatrix}$; mag $13{\cdot}6\ \text{ms}^{-1}$, dirn 2·09 radians

d Magnitude $2{\cdot}85\ \text{ms}^{-2}$; direction 4·71 radians

8 a i $x = \dfrac{e^t + e^{-t}}{2} + c_1; y = \dfrac{e^t - e^{-t}}{2} + c_2$ **ii** $(1, 0)$

b i $\sqrt{\dfrac{e^{2t} + e^{-2t}}{2}}$ **ii** $\dfrac{e^{2t}+1}{e^{2t}-1}$

c $\begin{pmatrix} 74{\cdot}2 \\ 74{\cdot}2 \end{pmatrix}$

9 a $(1, 0)$ **b** $\begin{pmatrix} 1{\cdot}15 \\ -1{\cdot}75 \end{pmatrix}$

10 a In both cases the angle is $\tan^{-1}(-1)$ and in same quadrant.

b Expand both and use $\sin^2 t + \cos^2 t = 1$