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Unit 2 : Sequences and Series - Lesson 10

Finite Sums

LI

- Simplify finite sums as algebraic expressions.

SC

- Algebra.

Rules for Finite Sums

In the following, $r, a, p, n \in \mathbb{N}$ and $k \in \mathbb{R}$

$$(1) \quad \sum_{r=a}^n k f(r) = k \sum_{r=a}^n f(r)$$

$$(2) \quad \sum_{r=p}^n f(r) = \sum_{r=1}^n f(r) - \sum_{r=1}^{p-1} f(r) \quad (1 < p < n)$$

$$(3) \quad \sum_{r=1}^n (f(r) + g(r)) = \sum_{r=1}^n f(r) + \sum_{r=1}^n g(r)$$

$$(4) \quad \sum_{r=1}^n f(r + 1) - \sum_{r=1}^n f(r) = f(n + 1) - f(1)$$

(Telescoping sum)

When $f(r)$ takes special forms, some important formulae result :

$$\underline{f(r) = 1 :} \quad \sum_{r=1}^n 1 = n$$

$$\underline{f(r) = r :} \quad \sum_{r=1}^n r = \frac{n(n + 1)}{2}$$

$$\underline{f(r) = r^2 :} \quad \sum_{r=1}^n r^2 = \frac{n(n + 1)(2n + 1)}{6}$$

$$\underline{f(r) = r^3 :} \quad \sum_{r=1}^n r^3 = \frac{n^2(n + 1)^2}{4} = \left(\sum_{r=1}^n r \right)^2$$

Example 1

Express $\sum_{r=1}^n (5r + 2)$ in the form $an^2 + bn$.

$$\sum_{r=1}^n (5r + 2) = \sum_{r=1}^n 5r + \sum_{r=1}^n 2$$

$$\Rightarrow \sum_{r=1}^n (5r + 2) = 5 \sum_{r=1}^n r + 2 \sum_{r=1}^n 1$$

$$\Rightarrow \sum_{r=1}^n (5r + 2) = \frac{5n(n+1)}{2} + 2n$$

$$\Rightarrow \sum_{r=1}^n (5r + 2) = \frac{5n^2}{2} + \frac{5n}{2} + \frac{4n}{2}$$

$$\Rightarrow \boxed{\sum_{r=1}^n (5r + 2) = \frac{5}{2}n^2 + \frac{9}{2}n}$$

Example 2

Express $\sum_{r=1}^n ((r + 1)^3 - r^3)$ in the form $n p(n)$,

stating the quadratic $p(n)$.

This is a telescoping sum with $f(r) = r^3$; so,

$$\begin{aligned} \sum_{r=1}^n ((r + 1)^3 - r^3) &= (n + 1)^3 - 1^3 \\ \Rightarrow \sum_{r=1}^n ((r + 1)^3 - r^3) &= (n^3 + 3n^2 + 3n + 1) - 1 \end{aligned}$$

Binomial
Theorem

$$\begin{aligned} \Rightarrow \boxed{\sum_{r=1}^n ((r + 1)^3 - r^3)} &= n(n^2 + 3n + 3) \\ &(p(n) = n^2 + 3n + 3) \end{aligned}$$

Example 3

Express $\sum_{r=1}^n (12r^2 + 4r)$ in the form $k n(n + 1)^m$,

stating the values of the constants k and m .

$$\begin{aligned}\sum_{r=1}^n (12r^2 + 4r) &= \sum_{r=1}^n 12r^2 + \sum_{r=1}^n 4r \\ \Rightarrow \sum_{r=1}^n (12r^2 + 4r) &= 12 \sum_{r=1}^n r^2 + 4 \sum_{r=1}^n r \\ \Rightarrow \sum_{r=1}^n (12r^2 + 4r) &= \frac{12n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} \\ \Rightarrow \sum_{r=1}^n (12r^2 + 4r) &= 2n(n+1)(2n+1) + 2n(n+1) \\ \Rightarrow \sum_{r=1}^n (12r^2 + 4r) &= 2n(n+1)(2n+1+1) \\ \Rightarrow \sum_{r=1}^n (12r^2 + 4r) &= 2n(n+1)(2n+2) \\ \Rightarrow \sum_{r=1}^n (12r^2 + 4r) &= 4n(n+1)(n+1) \\ \Rightarrow \boxed{\sum_{r=1}^n (12r^2 + 4r) = 4n(n+1)^2} \quad (k = 4, m = 2)\end{aligned}$$

AH Maths - MiA (2nd Edn.)

- pg. 168-170 Ex. 10.1
Q 4, 8, 9, 11.
- pg. 172-3 Ex. 10.2
Q 1, 2, 4, 5.

Ex. 10.1

- 4** Express as a function of n

a $\sum_{r=1}^n 2r$

b $\sum_{r=1}^n -1$

c $\sum_{r=1}^n (6r - 1)$

d $\sum_{r=1}^n (4 - 3r)$

e $\sum_{r=1}^{2n} (r - 8)$

- 8** Find the value of the unknown in each case.

a $\sum_{r=1}^n (2r + 3) = 117$

b $\sum_{r=1}^n (4 - 5r) = -148$

- 9** Use the fact that $\sum_{r=1}^n f(r+1) - \sum_{r=1}^n f(r) = f(n+1) - f(1)$ to help you express each sum without the aid of sigma notation explicitly as a function of n .

a $\sum_{r=1}^n ((r+1)^2 - r^2)$

Hint: $\sum_{r=1}^n ((r+1)^2 - r^2) = \sum_{r=1}^n (r+1)^2 - \sum_{r=1}^n r^2$

b $\sum_{r=1}^n \sqrt{r+1} - \sqrt{r}$

c $\sum_{r=1}^n \sin(r+1) - \sin r$

d $\sum_{r=1}^n \left(\frac{1}{r+1} - \frac{1}{r}\right)$

e $\sum_{r=2}^n \left(\frac{1}{r} - \frac{1}{r-1}\right)$

f $\sum_{r=2}^n \left(\frac{1}{r-1} - \frac{1}{r}\right)$

- 11** a Express $\ln\left(1 + \frac{1}{n}\right)$ in the form $\ln\left(\frac{a}{b}\right)$.

- b Hence express it in the form $\ln a - \ln b$.

- c Express $\sum_{r=1}^n \ln\left(1 + \frac{1}{r}\right)$ as a function of n without sigma notation.

- d Evaluate $\sum_{r=1}^9 \log_{10}\left(1 + \frac{1}{r}\right)$.

- e Find the value of $\log_{10}\left(2 \times 1\frac{1}{2} \times 1\frac{1}{3} \times 1\frac{1}{4} \times 1\frac{1}{5} \times \dots \times 1\frac{1}{99}\right)$.

Ex. 10.2

- 1** Express each of these as a function of n without the use of the sigma notation.

a $\sum_{r=1}^n 3r^2$

b $\sum_{r=1}^n (2r^2 + 3)$

c $\sum_{r=1}^n (r^2 + 3r - 1)$

d $\sum_{r=1}^n (5r^2 - 3r - 2)$

e $\sum_{r=1}^n (4 - 2r - 3r^2)$

f $\sum_{r=1}^n (r - 2)(r + 1)$

- 2** Evaluate

a $\sum_{r=1}^7 2r^2$

b $\sum_{r=1}^{10} (r^2 + 5)$

c $\sum_{r=1}^{20} (r^2 - 4r)$

d $\sum_{r=1}^{100} (2 - 3r - r^2)$

e $\sum_{r=1}^{12} (r + 4)(r - 3)$

f $\sum_{r=1}^9 (2r - 1)(r - 1)$

- 4** Express each of these as a function of n without the use of the sigma notation.

a $\sum_{r=1}^n 4r^3$

b $\sum_{r=1}^n (r^3 + 3r)$

c $\sum_{r=1}^n (2r^3 + 4r - 3)$

d $\sum_{r=1}^n (r^3 + r^2 + r + 1)$

e $\sum_{r=1}^n r^2(r - 1)$

f $\sum_{r=1}^n (r - 3)(r - 1)^2$

- 5** Evaluate

a $\sum_{r=1}^5 5r^3$

b $\sum_{r=1}^{10} (r^3 + r)$

c $\sum_{r=1}^8 (r^3 - 2r - 1)$

d $\sum_{r=1}^{50} r(r + 1)(r + 2)$

e $\sum_{r=1}^{12} (r + 4)(r - 3)$

f $\sum_{r=1}^{25} r(r - 1)(r + 1)$

Answers to AH Maths (MiA), pg. 168-70, Ex. 10.1

4 a $n(n + 1)$

b $-n$

c $3n(n + 1) - n$

d $4n - \frac{3}{2}n(n + 1)$

e $n(2n + 1) - 16n$

8 a $n = 9$

b $n = 8$

9 a $(n + 1)^2 - 1$

b $\sqrt{n + 1} - 1$

c $\sin(n + 1) - \sin 1$

d $\frac{1}{n + 1} - 1$

e $\frac{1}{n} - 1$

f $1 - \frac{1}{n}$

11 a $\ln\left(\frac{n + 1}{n}\right)$ b $\ln(n + 1) - \ln n$ c $\ln(n + 1)$

d 1

e 2

Answers to AH Maths (MiA), pg. 172-3, Ex. 10.2

1 a $\frac{n}{2}(n + 1)(2n + 1)$

b $\frac{n}{3}(n + 1)(2n + 1) + 3n$

c $\frac{n}{6}(n + 1)(2n + 1) + \frac{3n}{2}(n + 1) - n$

d $\frac{5n}{6}(n + 1)(2n + 1) - \frac{3n}{2}(n + 1) - 2n$

e $-\frac{n}{2}(n + 1)(2n + 1) - n(n + 1) + 4n$

f $\frac{n}{6}(n + 1)(2n + 1) - \frac{n}{2}(n + 1) - 2n$

2 a 280

b 435

c 2030

d -353 300

e 584

f 444

4 a $4\left[\frac{n}{2}(n + 1)\right]^2$

b $\left[\frac{n}{2}(n + 1)\right]^2 + \frac{3n}{2}(n + 1)$

c $2\left[\frac{n}{2}(n + 1)\right]^2 + 2n(n + 1) - 3n$

d $\left[\frac{n}{2}(n + 1)\right]^2 + \frac{n}{6}(n + 1)(2n + 1) + \frac{n}{2}(n + 1) + n$

e $\left[\frac{n}{2}(n + 1)\right]^2 - \frac{n}{6}(n + 1)(2n + 1)$

f $\left[\frac{n}{2}(n + 1)\right]^2 - \frac{5n}{6}(n + 1)(2n + 1) + \frac{7n}{2}(n + 1) - 3n$

5 a 1125

b 3080

c 1216

d 1756950

e 584

f 105 300