23 / 8 / 17

Unit 1 : Differential Calculus - Lesson 9

Parametric Differentiation

LI

- Differentiate functions parametrically.

SC

- Formulae.

A curve $y = f(x)$ is defined parametrically if x and y are written in terms of an independent variable called the parameter (usually denoted by t or θ). It often helps to think of the parameter as representing time.

The functions $x(t)$ and $y(t)$ are viewed as 2 functions of t and are called parametric functions or the parametric equations of the curve.

If the parameter can be eliminated, then an implicit equation (called the constraint equation) can be formed; this only contains x and y.

As an example, consider the parametric equations,

$$x(t) = 3\cos t, \quad y(t) = 3\sin t$$

Squaring, adding and simplifying gives (suppressing the dependence on the parameter t),

$$x^2 + y^2 = 9$$

This constraint equation describes a circle with centre the origin and radius 3. So, the parametric equations above for $x(t)$ and $y(t)$ describe this circle.

Each value of t gives a coordinate (x, y). So, varying t traces out a circle. Or, think of an object moving along the circle as t varies. Starting at $t = 0$, we are at $(3, 0)$; at $t = \pi/2$, we are at $(0, 3)$; at $t = \pi$, we are at $(-3, 0)$; at $t = 3\pi/2$, we are at $(0, -3)$. At $t = 2\pi$, we are back at $(3, 0)$.

Parametric differentiation is a technique for finding derivatives of a parametrically defined curve.

First Derivative of Parametric Functions

Given a parametrically defined curve $y = f(x)$ with parameter t, the first parametric derivative is :

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

This looks awkward in the above Leibniz notation, so the Newton Dot Notation is often used :

$$\boxed{\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}}$$

$\dot{y}$ means differentiate y wrt to t, i.e. $\dfrac{dy}{dt}$.

$\dot{x}$ means differentiate x wrt to t, i.e. $\dfrac{dx}{dt}$.

Second Derivative of Parametric Functions

Given a parametrically defined curve $y = f(x)$ with parameter t, the second parametric derivative is :

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

Again, Dot Notation is extremely advantageous :

$$\boxed{\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}}$$

$\ddot{y}$ means differentiate y twice wrt to t, i.e. $\frac{d^2y}{dt^2}$.

$\ddot{x}$ means differentiate x twice wrt to t, i.e. $\frac{d^2x}{dt^2}$.

Example 1

Show that the point $A\ (0, 3)$ lies on the curve defined parametrically by the equations,

$$x = t - \frac{1}{t}, \quad y = 2t + \frac{1}{t}$$

To show that the point $A\ (0, 3)$ lies on the curve, it must be shown that there is a common t - value that satisfies both equations.

We thus solve the equations $x(t) = 0$ and $y(t) = 3$ for t.

$\underline{x(t) = 0:}$

$$t - \frac{1}{t} = 0$$

$$\Rightarrow \quad t^2 - 1 = 0$$

$$\Rightarrow \quad \underline{t = \pm 1}$$

$\underline{y(t) = 3:}$

$$2t + \frac{1}{t} = 3$$

$$\Rightarrow \quad 2t^2 + 1 = 3t$$

$$\Rightarrow \quad 2t^2 - 3t + 1 = 0$$

$$\Rightarrow \quad (2t - 1)(t - 1) = 0$$

$$\Rightarrow \quad \underline{t = 1/2, 1}$$

As there is a common t - value ($t = 1$) satisfying $x(t) = 0$ and $y(t) = 3$, $A\ (0, 3)$ lies on the curve.

Mar 1-14:16

Example 2

Find the gradient of the tangent line to the curve,

$$x = 2\theta - \sin\theta \quad , \quad y = 4\theta + \cos\theta$$

at $\theta = \pi/6$.

$$\boxed{\begin{array}{ll} x = 2\theta - \sin\theta \quad , & y = 4\theta + \cos\theta \\ \dot{x} = 2 - \cos\theta \quad , & \dot{y} = 4 - \sin\theta \end{array}}$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

$$\therefore \quad \underline{\frac{dy}{dx} = \frac{4 - \sin\theta}{2 - \cos\theta}}$$

When $\theta = \pi/6$,

$$\dot{x}(\pi/6) = 2 - \cos(\pi/6) = \underline{2 - \sqrt{3}/2}$$

$$\dot{y}(\pi/6) = 4 - \sin(\pi/6) = 4 - 1/2 = \underline{7/2}$$

Hence,

$$\left(\frac{dy}{dx}\right)_{\theta = \pi/6} = \frac{7/2}{2 - \sqrt{3}/2}$$

$$\Rightarrow \boxed{\left(\frac{dy}{dx}\right)_{\theta = \pi/6} = \frac{7}{4 - \sqrt{3}}}$$

Example 3

Show that there is only one stationary point on the curve,

$$x = 5 + 4t \ , \ y = 3 - 3t^2$$

and find its coordinates. Show also that the second derivative is constant.

$$\boxed{\begin{aligned} x &= 5 + 4t \ , & y &= 3 - 3t^2 \\ \dot{x} &= 4 \ , & \dot{y} &= -6t \\ \ddot{x} &= 0 \ , & \ddot{y} &= -6 \end{aligned}}$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-6t}{4}$$

$$\Rightarrow \quad \underline{\frac{dy}{dx} = -\frac{3t}{2}}$$

For stationary points,

$$\frac{dy}{dx} = 0$$

$$\therefore \quad -\frac{3t}{2} = 0$$

$$\Rightarrow \quad \underline{t = 0}$$

As there is only 1 solution to $\frac{dy}{dx} = 0$, there is only 1 SP.

$$x(t) = 5 + 4t \Rightarrow \underline{x(0) = 5}$$

$$y(t) = 3 - 3t^2 \Rightarrow \underline{y(0) = 3}$$

$\therefore$ Stationary point coordinates : (5, 3)

Mar 1-14:16

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{(4)(-6) - (0)(-6)}{4^3}$$

$$\Rightarrow \quad \boxed{\frac{d^2y}{dx^2} = -\frac{3}{8} = \text{constant}}$$

Jul 3-21:22

AH Maths - MiA (2nd Edn.)

- pg. 96-7 Ex. 6.8 Q 1 a - d, 2 a, 3 a, b, 4 a, 8.

Ex. 6.8

1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the curve defined by each pair of parametric equations.

a $x = t$, $y = \frac{1}{t}$

b $x = t^2$, $y = \ln t$

c $x = t + \sin t$, $y = t - \cos t$

d $x = 3t^3 - t$, $y = 4t^2$

2 A curve is defined by the equations $x = t^2 + \frac{2}{t}$ and $y = t^2 - \frac{2}{t}$.

a Find the coordinates of the turning point on the curve.

3 a Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the curve defined by $x = t^2 - \frac{1}{t^2}$ and $y = t^2 + \frac{1}{t^2}$.

b Determine the coordinates of the turning point on the curve.

4 For the curve defined by $x = \frac{2t}{1 - t^2}$, $y = \frac{1 + t^2}{1 - t^2}$ show that

a $\frac{dy}{dx} = \frac{x}{y}$

8 $x = 1 + \sin^2\theta$ and $y = 1 - \sec^2\theta$ are the parametric representations of a curve. Show that, at the point where $\tan\theta = 2$, the equation of the tangent is $25x + y = 41$.

Answers to AH Maths (MiA), pg. 96-7, Ex. 6.8

1 **a** $-\dfrac{1}{t^2}, \dfrac{2}{t^3}$ **b** $\dfrac{1}{2t^2}, -\dfrac{1}{2t^4}$

c $\dfrac{1+\sin t}{1+\cos t}, \dfrac{(\cos t+\sin t+1)}{(1+\cos t)^3}$

d $\dfrac{8t}{9t^2-1}, \dfrac{-8-72t^2}{(9t^2-1)^3}$

2 **a** $(-1, 3)$

3 **a** $\dfrac{t^4-1}{t^4+1}, \dfrac{4t^6}{(t^4+1)^3}$ **b** $(0, 2)$

4 **a** $\dfrac{dx}{dt} = \dfrac{2+2t^2}{(1-t^2)^2}; \dfrac{dy}{dt} = \dfrac{4t}{(1-t^2)^2}; \dfrac{dy}{dx} = \dfrac{2t}{1+t^2} = \dfrac{x}{y}$

8 $\dfrac{dy}{dx} = -\dfrac{1}{\cos^4\theta}$. When $\tan\theta = 2$, $\cos\theta = \frac{1}{\sqrt{5}}$, $\sin\theta = \frac{2}{\sqrt{5}}$