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Force, Energy and Periodic Motion - Lesson 1

Circular Motion Theory

LI

- Know and use the definitions of Angular Displacement and Angular Velocity.
- Express linear velocity and linear acceleration in Polar Coordinates.

SC

- Vectors.
- Product Rule.

Jan 10-12:11

Definition :

Circular motion is motion in a circle of fixed radius.

Circular Motion is best described by using the Polar Coordinate system (instead of the x-y Cartesian Coordinate system).

Polar Coordinates

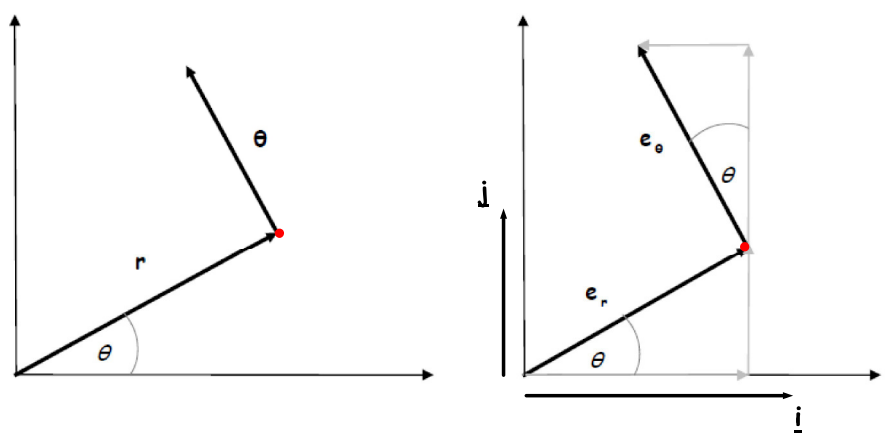
Definition :

The **polar coordinate system** is described by **polar coordinates** (r, θ), where $r > 0$ is the **radial coordinate** (aka **radius**) and $\theta \in (-\pi, \pi]$ is the **angular coordinate** (aka **polar angle**).

θ is measured from the positive x-axis direction and $-\pi < \theta \leq \pi$.

Radial and Tangential Unit Vectors

Instead of using the unit vectors $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$, we use a different pair of perpendicular unit vectors $\underline{\mathbf{e}}_r$ and $\underline{\mathbf{e}}_\theta$.



Definition :

The **(radial) displacement** of a particle in polar coordinates is,

$$\mathbf{r} = r\,\mathbf{e}_r$$

where $\mathbf{e}_r$ is the **radial unit vector**.

Definition :

The **angular displacement** of a particle in polar coordinates is,

$$\boldsymbol{\theta} = \theta\,\mathbf{e}_\theta$$

where $\mathbf{e}_\theta$ is the **tangential unit vector** (aka **angular unit vector**).

Important Fact

For circular motion, the $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ unit vectors are fixed (constant), but the $\underline{\mathbf{e}}_r$ and $\underline{\mathbf{e}}_\theta$ vectors change (not constant).

The following 2 lemmas (and their respective corollaries) follow from the previous diagram and the definitions thereafter.

Lemma :

The radial unit vector can be written in terms of Cartesian unit vectors as,

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

Corollary :

The displacement of a particle in Cartesian coordinates is,

$$\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$$

Lemma :

The tangential unit vector can be written in terms of Cartesian unit vectors as,

$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

Corollary :

The angular displacement of a particle in Cartesian coordinates is,

$$\boldsymbol{\theta} = (-\theta\sin\theta)\,\mathbf{i} + (\theta\cos\theta)\,\mathbf{j}$$

As we are considering circular motion, r is constant, but θ is a function of time : θ (t)

Kinematical Quantities for Rotational Motion

Definition :

The **angular velocity** is the time-derivative of angular displacement,

omega : ω

$$\boldsymbol{\omega} \equiv \dot{\boldsymbol{\theta}} \overset{def}{=} \frac{d\,\boldsymbol{\theta}}{dt}$$

Angular velocity has derived SI unit the rad s^{-1}.

Magnitude of angular velocity $\underline{\boldsymbol{\omega}}$ is angular speed ω.

In this course, we take $\boldsymbol{\omega}$ to be constant. Then the relation $\omega = \frac{d\theta}{dt}$ simplifies.

Corollary :

For constant angular velocity,

$$\theta = \omega t$$

In the special case of 1 revolution, the result in the previous corollary becomes the content of the following corollary.

Corollary :

For 1 revolution, the period and angular speed are linked by,

$$T = \frac{2\pi}{\omega}$$

(To see this, take $\theta = 2\pi$ radians and t as the period T)

Connecting Linear Speed (v) to Angular Speed (ω)

Differentiating $\underline{\mathbf{r}} = (r\cos\theta)\,\underline{\mathbf{i}} + (r\sin\theta)\,\underline{\mathbf{j}}$ wrt t and remembering that r is constant but θ depends on t, we have :

$$\underline{\mathbf{r}} = (r\cos\theta)\,\underline{\mathbf{i}} + (r\sin\theta)\,\underline{\mathbf{j}}$$

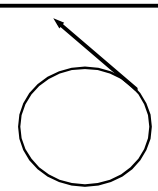
$$\therefore \quad \underline{\mathbf{v}} = (-r\dot{\theta}\sin\theta)\,\underline{\mathbf{i}} + (r\dot{\theta}\cos\theta)\,\underline{\mathbf{j}}$$

$$\Rightarrow \quad \underline{\mathbf{v}} = r\dot{\theta}\,(-\sin\theta\,\underline{\mathbf{i}} + \cos\theta\,\underline{\mathbf{j}})$$

$$\Rightarrow \quad \boxed{\underline{\mathbf{v}} = r\dot{\theta}\,\underline{\mathbf{e}}_\theta = \omega r\,\underline{\mathbf{e}}_\theta}$$

$$\therefore \quad \boxed{v = \omega r} \qquad \text{(Taking magnitudes)}$$

The linear velocity (v) has magnitude ωr and is tangent to the circle (points in same direction as $\underline{\mathbf{e}}_\theta$)



Definition :

The **angular acceleration** is the time-derivative of angular velocity,

alpha : α

$$\boldsymbol{\alpha} \overset{def}{=} \frac{d\,\boldsymbol{\omega}}{dt} \equiv \dot{\boldsymbol{\omega}} = \frac{d^2\,\boldsymbol{\theta}}{dt^2}$$

Angular acceleration has derived SI unit the rad s^{-2}.

Magnitude of angular acceleration $\underline{\alpha}$ is α.

In this course, we normally take $\boldsymbol{\alpha}$ to be 0.

Connecting Linear Acceleration (a) to Angular Speed (ω)

Differentiating $\underline{\mathbf{v}} = (-r\dot{\theta}\sin\theta)\,\underline{\mathbf{i}} + (r\dot{\theta}\cos\theta)\,\underline{\mathbf{j}}$ wrt t, we have, using the Product Rule (as both θ and it's derivative are functions of time) :

$$\underline{\mathbf{v}} = (-r\dot{\theta}\sin\theta)\,\underline{\mathbf{i}} + (r\dot{\theta}\cos\theta)\,\underline{\mathbf{j}}$$

$$\therefore\ \underline{\mathbf{a}} = -r(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\,\underline{\mathbf{i}} + r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\,\underline{\mathbf{j}}$$

As we take $\omega = \dot{\theta}$ to be constant, this means that $\ddot{\theta} = 0$. So,

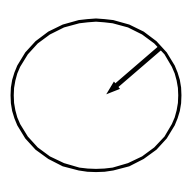
$$\underline{\mathbf{a}} = -r\dot{\theta}^2\cos\theta\,\underline{\mathbf{i}} - r\dot{\theta}^2\sin\theta\,\underline{\mathbf{j}}$$

$$\Rightarrow\ \underline{\mathbf{a}} = -r\dot{\theta}^2(\cos\theta\,\underline{\mathbf{i}} + \sin\theta\,\underline{\mathbf{j}})$$

$$\Rightarrow\ \boxed{\underline{\mathbf{a}} = -r\dot{\theta}^2\,\underline{\mathbf{e}}_r = -\omega^2 r\,\underline{\mathbf{e}}_r}$$

$$\therefore\ \boxed{a = \omega^2 r = \frac{v^2}{r}} \quad (\text{using } v = \omega r)$$

The linear acceleration (a) has magnitude $\omega^2 r$ and is towards the centre of the circle (points in direction opposite to $\underline{\mathbf{e}}_r$)



Example 1

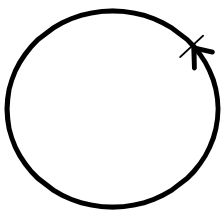
A uniformly rotating disc completes 1 revolution every 20 s.

Find the angular speed of the disc in :

(a) rev min^{-1}

(b) rad min^{-1}

(c) rad s^{-1}.



1 rev. in 20 s
covers 2π radians

$\boxed{1 \text{ rev} = 2\pi \text{ rad}}$

$\boxed{1 \text{ min} = 60 \text{ s}}$

(a) As the rotation is uniform, ω is constant. 1 rev is completed every 20 s = 1/3 min, so, using $\omega = \theta/t$,

$$\omega = \frac{1 \text{ rev}}{1/3 \text{ min}} \Rightarrow \boxed{\omega = 3 \text{ rev min}^{-1}}$$

(b) 1 rev is equivalent to 2π rad, so,

$$\omega = \frac{2\pi \text{ rad}}{1/3 \text{ min}} \Rightarrow \boxed{\omega = 6\pi \text{ rad min}^{-1}}$$

(c) As 1/3 min = 20 s, we have,

$$\omega = \frac{2\pi \text{ rad}}{20 \text{ s}} \Rightarrow \boxed{\omega = \pi/10 \text{ rad s}^{-1}}$$

Example 2

Find the speed in $m\ s^{-1}$ of a particle moving on a circular path of radius 30 cm with an angular speed of $5\ rev\ min^{-1}$.

$$\boxed{1\ rev = 2\pi\ rad}$$

$$\boxed{1\ min = 60\ s}$$

An angular speed of $5\ rev\ min^{-1}$ means,

$$\omega = \frac{5\ rev}{1\ min}$$

$$\Rightarrow \quad \omega = \frac{5\ (2\pi)\ rad}{60\ s}$$

$$\Rightarrow \quad \underline{\omega = \pi/6\ rad\ s^{-1}}$$

$$v = \omega r$$

$$\therefore \quad v = (\pi/6)\ (0\ .\ 3)$$

$$\Rightarrow \quad v = (\pi/6)\ (3/10)$$

$$\Rightarrow \quad \boxed{v = \pi/20\ m\ s^{-1}}$$

Example 3

Find the acceleration of a particle moving in a circular path of radius 60 cm with a constant angular velocity of 4 rad s^{-1}.

$$a = \omega^2 r$$

$$\therefore \quad a = 4^2 (0.6)$$

$$\Rightarrow \quad a = 16 (6/10)$$

$$\Rightarrow \quad \boxed{a = 48/5 \text{ m s}^{-2}}$$

Sep 16-10:04

Blue Book

- pg. 307-308 Ex. 13 A All Q.
- pg. 313 Ex. 13 B Q 1 - 3.