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Unit 1 : Differential Calculus - Lesson 8

Logarithmic Differentiation

LI

- Differentiate functions logarithmically.

SC

- Log. Rules.
- Chain Rule.

Logarithmic differentiation is a technique for differentiating functions by taking logarithms first

Logarithmic differentiation should be used when any one of the following indicators are present.

- Bracketed terms with fractional powers.
- Variable is in the power.
- Product or quotient of more than 2 functions.

Example 1

Differentiate $y = 3^x$.

$$y = 3^x$$

$$\therefore \quad \ln y = x \ln 3$$

$$\Rightarrow \quad \ln y = (\ln 3)\, x$$

$$\therefore \quad y'/y = \ln 3$$

$$\Rightarrow \quad y' = y \ln 3$$

$$\Rightarrow \quad \boxed{\begin{array}{c} y' = 3^x . \ln 3 \\ \left(y' = (\ln 3)\, 3^x \right) \end{array}}$$

Example 2

Differentiate $f(x) = (\sin x)^x$.

$$f(x) = (\sin x)^x$$

$$\therefore \quad \ln f(x) = \ln (\sin x)^x$$

$$\Rightarrow \quad \ln f(x) = x \ln (\sin x)$$

$$\therefore \quad f'(x)/f(x) = 1 . \ln (\sin x) + x . (\cos x)/\sin x$$

$$\Rightarrow \quad f'(x)/f(x) = \ln (\sin x) + x \cot x$$

$$\Rightarrow \quad f'(x) = f(x) (\ln (\sin x) + x \cot x)$$

$$\Rightarrow \quad \boxed{f'(x) = (\sin x)^x (\ln (\sin x) + x \cot x)}$$

Example 3

Find w' if $w = x^{\text{cosec } 4x}$.

$$w = x^{\text{cosec } 4x}$$

$$\therefore \quad \ln w = \ln\left(x^{\text{cosec } 4x}\right)$$

$$\Rightarrow \quad \ln w = \text{cosec } 4x \, . \, \ln x$$

$$\therefore \quad w'/w = -4 \text{ cosec } 4x \cot 4x \, . \, \ln x + \text{cosec } 4x \, . \, (1/x)$$

$$\Rightarrow \quad w' = w(-4 \text{ cosec } 4x \cot 4x \ln x + \text{cosec } 4x \, . \, (1/x))$$

$$\Rightarrow \quad w' = x^{\text{cosec } 4x}(-4 \text{ cosec } 4x \cot 4x \ln x + \text{cosec } 4x \, . \, (1/x))$$

$$\Rightarrow \quad \boxed{w' = \text{cosec } 4x \, . \, x^{\text{cosec } 4x}(-4 \cot 4x \ln x + 1/x)}$$

Example 4

Find the derivative of the function defined by $y(x) = (\cos x)^{\cot x}$.

$$y = (\cos x)^{\cot x}$$

$$\therefore \quad \ln y = \ln (\cos x)^{\cot x}$$

$$\Rightarrow \quad \ln y = \cot x \, . \, \ln (\cos x)$$

$$\therefore \quad y'/y = -\operatorname{cosec}^2 x \, . \, \ln (\cos x) + \cot x \, . \, ((-\sin x)/\cos x)$$

$$\Rightarrow \quad y'/y = -\operatorname{cosec}^2 x \, . \, \ln (\cos x) - \cot x \tan x$$

$$\Rightarrow \quad y'/y = -\operatorname{cosec}^2 x \, . \, \ln (\cos x) - 1$$

$$\Rightarrow \quad y' = -y(\operatorname{cosec}^2 x \, . \, \ln (\cos x) + 1)$$

$$\Rightarrow \quad \boxed{y' = -(\cos x)^{\cot x}(\operatorname{cosec}^2 x \, . \, \ln (\cos x) + 1)}$$

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Example 5

Differentiate $y = \dfrac{e^{4x} \sec x}{x^3}$.

$$y = \frac{e^{4x} \sec x}{x^3}$$

$$\therefore \quad \ln y = \ln\left(\frac{e^{4x} \sec x}{x^3}\right)$$

$$\Rightarrow \quad \ln y = \ln(e^{4x}) + \ln(\sec x) - \ln(x^3)$$

$$\Rightarrow \quad \ln y = 4x \ln(e) + \ln(\sec x) - 3 \ln x$$

$$\Rightarrow \quad \ln y = 4x + \ln(\sec x) - 3 \ln x$$

$$\therefore \quad y'/y = 4 + ((\sec x \tan x)/\sec x) - 3/x$$

$$\Rightarrow \quad y'/y = 4 + \tan x - 3/x$$

$$\Rightarrow \quad y' = y(4 + \tan x - 3/x)$$

$$\Rightarrow \quad \boxed{y' = \left(\frac{e^{4x} \sec x}{x^3}\right)(4 + \tan x - 3/x) \quad \left(y' = \frac{e^{4x} \sec x\,(4x + x \tan x - 3)}{x^4}\right)}$$

Example 6

Differentiate $y = \dfrac{\sqrt[3]{3x+4}\,\sqrt{2x+1}}{\sqrt[4]{4x-7}}$.

$$y = (3x+4)^{1/3}(2x+1)^{1/2}(4x-7)^{-1/4}$$

$$\therefore \ln y = \ln(3x+4)^{1/3} + \ln(2x+1)^{1/2} + \ln(4x-7)^{-1/4}$$

$$\Rightarrow \ln y = (1/3)\ln(3x+4) + (1/2)\ln(2x+1) - (1/4)\ln(4x-7)$$

$$\therefore y'/y = \frac{(1/3)(3)}{3x+4} + \frac{(1/2)(2)}{2x+1} - \frac{(1/4)(4)}{4x-7}$$

$$\Rightarrow y'/y = \frac{1}{3x+4} + \frac{1}{2x+1} - \frac{1}{4x-7}$$

$$\Rightarrow y' = y\left(\frac{1}{3x+4} + \frac{1}{2x+1} - \frac{1}{4x-7}\right)$$

$$\Rightarrow \boxed{y' = \left(\frac{\sqrt[3]{3x+4}\,\sqrt{2x+1}}{\sqrt[4]{4x-7}}\right)\left(\frac{1}{3x+4} + \frac{1}{2x+1} - \frac{1}{4x-7}\right)}$$

AH Maths - MiA (2^{nd} Edn.)

- pg. 92 Ex. 6.6 Q 1, 2, 4, 5, 7.

Ex. 6.6

1 Find the derivatives of these functions.

a $f(x) = 5^{2x}$ b $f(x) = (x+1)^{x-1}$ c $f(x) = e^{\sin^2 x}$

d $f(x) = 3^{e^x}$ e $f(x) = (\cos x)^x$

2 Find the derived function for each of these functions.

a $f(x) = x^{x^2}$ b $f(x) = \pi^{x^3}$ c $f(x) = \dfrac{e^x \sin x}{x}$

d $f(x) = xe^{-x}\cos x$ e $f(x) = (1-x^3)^{\sin x}$

4 Show that $y = e^{\cos^2 x} \Rightarrow \dfrac{d^2y}{dx^2} = (\sin^2 2x - 2\cos 2x)y$.

5 Show that the tangent to the curve with equation $y = (\sin x)^x$ at $x = \frac{\pi}{2}$ is parallel to the x-axis.

7 Find the equation of the tangent to the curve with equation $y = \dfrac{x(2x+1)^{\frac{3}{2}}}{(3x-4)^{\frac{2}{3}}}$ at the point where $x = 4$.

Answers to AH Maths (MiA), pg. 92, Ex. 6.6

1 **a** $(2\ln 5)5^{2x}$

b $(x+1)^{x-1}\left(\left(\dfrac{x-1}{x+1}\right)+\ln(x+1)\right)$

c $\sin 2x\, e^{\sin^2 x}$ **d** $\ln 3\, e^x\, 3^{e^x}$

e $(\ln\cos x - x\tan x)(\cos x)^x$

2 **a** $(x+2x\ln x)x^{x^2}$ **b** $3x^2\ln\pi\,.\,\pi^{x^3}$

c $\dfrac{e^x\left((x-1)\sin x + x\cos x\right)}{x^2}$

d $(\cos x - x\sin x - x\cos x)e^{-x}$

e $(1-x^3)^{\sin x}\left(\cos x\ln(1-x^3)-\dfrac{3x^2\sin x}{1-x^3}\right)$

4 $\dfrac{dy}{dx} = -2\cos x\sin x\,.\,e^{\cos^2 x} = -\sin 2x\, e^{\cos^2 x}$

$$\Rightarrow \frac{d^2y}{dx^2} = -\sin 2x\,.\,e^{\cos^2 x}\,.\,-2\cos x\sin x + e^{\cos^2 x}\,.\,-2\cos 2x = (\sin^2 2x - 2\cos 2x)\,e^{\cos^2 x}$$

5 $\ln y = x\ln(\sin x) \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = x\dfrac{\cos x}{\sin x} + \ln(\sin x) \Rightarrow$

$\dfrac{dy}{dx} = x\cos x\,(\sin x)^{x-1} + (\sin x)^x\ln(\sin x) = 0$ when $x = \dfrac{\pi}{2}$

7 $y = 9x - 9$

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