Trigonometric Graphs

**LI**
- Sketch graphs of related trigonometric functions.
- Know that the order in which functions are transformed sometimes matters.

**SC**
- Know the standard transformations (shifts and scalings).
To transform trigonometric graphs, see how the following points transform:

For Sine

- \((0^\circ, 0)\)
- \((90^\circ, 1)\)
- \((180^\circ, 0)\)
- \((270^\circ, -1)\)
- \((360^\circ, 0)\)

For Cosine

- \((0^\circ, 1)\)
- \((90^\circ, 0)\)
- \((180^\circ, -1)\)
- \((270^\circ, 0)\)
- \((360^\circ, 1)\)

\[y = \sin x\]

\[y = \cos x\]

For \(M \sin (b \times x)^\circ \) or \(M \cos (b \times x)^\circ\):

- \(M\) is the amplitude.
- \(b\) is the number of whole cycles between \(0^\circ\) and \(360^\circ\).
Example 1

Sketch the graph of \( y = 3 \cos (2x - 30^\circ) + 2 \), where \( 0 \leq x \leq 180 \), indicating where it cuts the \( y \)-axis.

The graph of \( y = 3 \cos (2x - 30^\circ) + 2 \) has 2 complete cycles in \( 0 \leq x \leq 360 \), so there is one complete cycle in \( 0 \leq x \leq 180 \).

To see how the standard coordinates change, rewrite as,

\[
y = 3 \cos (2x - 30^\circ) + 2 \\
= 3 \cos (2(x - 15))^\circ + 2
\]

\((0^\circ, 1) \rightarrow (0^\circ/2 + 15^\circ, 1 \times 3 + 2) = (15^\circ, 5)\)

\((90^\circ, 0) \rightarrow (90^\circ/2 + 15^\circ, 0 \times 3 + 2) = (60^\circ, 2)\)

\((180^\circ, -1) \rightarrow (90^\circ + 15^\circ, -1 \times 3 + 2) = (105^\circ, -1)\)

\((270^\circ, 0) \rightarrow (135^\circ + 15^\circ, 0 \times 3 + 2) = (150^\circ, 2)\)

\((360^\circ, 1) \rightarrow (180^\circ + 15^\circ, 1 \times 3 + 2) = (195^\circ, 5)\)

Since we are only considering the graph for \( 0 \leq x \leq 180 \), the last transformed coordinate above will not be included in the graph.

When \( x = 0 \), \( y = 3 \cos (-30)^\circ + 2 \approx 4.6 \).
Example 2

Sketch the graph of \( y = 40 \sin (10x - 60)° \), where \( 0 \leq x \leq 72 \), showing where the graph cuts the \( x \)-axis, and any stationary points.

The graph of \( y = 40 \sin (10x - 60)° \) has 10 complete cycles in \( 0 \leq x \leq 360 \), so there are two complete cycles in \( 0 \leq x \leq 72 \).

To see how the standard coordinates change, rewrite as,

\[
y = 40 \sin (10x - 60)° \\
= 40 \sin (10(x - 6))°
\]

\((0°, 0) \rightarrow (0°/10 + 6°, 0 \times 40) = (6°, 0)\)

\((90°, 1) \rightarrow (90°/10 + 6°, 1 \times 40) = (15°, 40)\)

\((180°, 0) \rightarrow (180°/10 + 6°, 0 \times 40) = (24°, 0)\)

\((270°, -1) \rightarrow (27° + 6°, -1 \times 40) = (33°, -40)\)

\((360°, 0) \rightarrow (360°/10 + 6°, 0 \times 40) = (42°, 0)\)

The remaining transformed coordinates can be obtained by symmetry of the graph and the above transformed coordinates. The \( x \)-coordinates increase by \( 9° \).

When \( x = 0 \), \( y = 40 \sin (-60)° \approx -34.6 \).
Example 3

The diagram shows the graph of
\[ y = P \sin (x + Q)^\circ + R. \]

Find the values of \( P, Q \) and \( R \).

The amplitude is \( (5 - (-1))/2 = 3 \). So, \( P = 3 \).

The maximum value of \( 3 \sin (x + Q)^\circ \) is 3; the maximum value of \( 3 \sin (x + Q)^\circ + R \) is 5 (from the graph). Hence, \( R = 2 \).

The maximum value of \( 3 \sin (x + Q)^\circ + 2 \) occurs when \( x = 60 \) (from the graph). The maximum value of \( \sin x^\circ \) occurs when \( x = 90 \). So, \( Q = 30 \).

\[ P = 3, \; Q = 30, \; R = 2 \]
Example 4

Given $f(x) = 2 \sin (3x - 60)^\circ + 5$, where $0 \leq x \leq 120$, find the greatest and least values of $f(x)$ and the values of $x$ for which they occur.

$f(x)$ is a maximum when $\sin (3x - 60)^\circ = 1$; so, the maximum value of $f(x)$ is $2(1) + 5 = 7$. This value occurs when $(3x - 60)^\circ = 90^\circ$, i.e. when $x = 50$.

Max. value of $f(x) = 7$ occurs when $x = 50$

$f(x)$ is a minimum when $\sin (3x - 60)^\circ = -1$; so the minimum value of $f(x)$ is $2(-1) + 5 = 3$. This value occurs when $(3x - 60)^\circ = 270^\circ$, i.e. when $x = 110$.

Min. value of $f(x) = 3$ occurs when $x = 110$
Example 5

Given $g(x) = 4 - 3 \cos(x + \pi/3)$, where $0 \leq x \leq 2\pi$, find the greatest value of $g(x)$ and the value of $x$ for which it occurs.

g(x) is a maximum when $\cos(x + \pi/3) = -1$; so, the maximum value of $g(x)$ is $4 - 3(-1) = 7$. This value occurs when $(x + \pi/3) = \pi$, i.e. when $x = 2\pi/3$.

Max. value of $g(x) = 7$ occurs when $x = 2\pi/3$
Example 6

Express \( \cos (30x) + \sqrt{3} \sin (30x) \) in the form \( k \cos (30x - \alpha) \), where \( k > 0 \) and \( 0 \leq \alpha \leq 360 \).

Hence sketch the graph of

\[
y = \cos (30x) + \sqrt{3} \sin (30x) + 1, \text{ where } 0 \leq x \leq 12.
\]

Using wave function theory, we have,

\[
\cos (30x) + \sqrt{3} \sin (30x) = 2 \cos (30x - 60^\circ)
\]

So, \( y = \cos (30x) + \sqrt{3} \sin (30x) + 1 \) can be written as,

\[
y = 2 \cos (30x - 60^\circ) + 1
\]

\[
= 2 \cos (30(x - 2)) + 1
\]

The graph of \( y = 2 \cos (30(x - 2)) + 1 \) has 30 complete cycles in \( 0 \leq x \leq 360 \), so there is one complete cycle in \( 0 \leq x \leq 12 \).

\[
(0^\circ, 1) \rightarrow (0/30 + 2^\circ, 1 \times 2 + 1) = (2^\circ, 3)
\]

\[
(90^\circ, 0) \rightarrow (90/30 + 2^\circ, 0 \times 2 + 1) = (5^\circ, 1)
\]

\[
(180^\circ, -1) \rightarrow (6^\circ + 2^\circ, -1 \times 2 + 1) = (8^\circ, -1)
\]

\[
(270^\circ, 0) \rightarrow (9^\circ + 2^\circ, 0 \times 2 + 1) = (11^\circ, 1)
\]

\[
(360^\circ, 1) \rightarrow (12^\circ + 2^\circ, 1 \times 2 + 1) = (14^\circ, 3)
\]

Since we are only considering the graph for \( 0 \leq x \leq 12 \), the last transformed coordinate above will not be included in the graph.
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