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Matrices and Systems of Equations - Lesson 7

Transformation Matrices

LI

- Know Basic Transformation Matrices.
- Find Image Points under a transformation.
- Find Invariant Points.
- Find Composite Transformations.
- Find Equations of Transformed Curves.

SC

- Matrix multiplication.
- 2 x 2 inverses.

A geometrical transformation is a way of changing points in space

A linear transformation in the plane is a function that sends a point $P(x, y)$ to $Q(ax + by, cx + dy)$ $(a, b, c, d \in \mathbb{R})$; the function is described by a 2×2 transformation matrix :

$$\underbrace{\begin{pmatrix} x' \\ y' \end{pmatrix}}_{Q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{P}$$

A transformation matrix is obtained by looking at where the geometrical transformation sends the points $(1, 0)$ and $(0, 1)$

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Basic Transformation Matrices

Geometrical Transformation	Transformation Matrix
Reflection in x - axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in y - axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Anticlockwise rotation of angle θ about origin	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Scaling (aka Dilatation)	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ $(k \in \mathbb{R})$ ($k > 1$: enlargement) ($k < 1$: reduction)

A composite transformation is a combination of at least 2 basic transformations; they are found by matrix multiplication

An invariant point is a point whose coordinates stay the same under a transformation

Example 1

Find the matrix associated with the transformation,
$x' = 2x + 5y, y' = x - 3y.$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x + 5y \\ x - 3y \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{\text{Transformation matrix : } \begin{pmatrix} 2 & 5 \\ 1 & -3 \end{pmatrix}}$$

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Example 2

Find the image of the point P (3, − 1) under the transformation associated with the matrix $\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$.

$$\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$

$\boxed{\text{Image of } P : \begin{pmatrix} 5 \\ 9 \end{pmatrix}}$

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Example 3

Find the transformation matrix associated with a reflection in the x - axis followed by an anticlockwise rotation of $90°$ about the origin.

The order in which the transformations are done is important, as the order in which matrices are multiplied is important. Let the reflection matrix be denoted by T_R and the rotation matrix by T_θ. The required matrix is $T_\theta T_R$, as T_R acts first on point (x, y), then T_θ acts on the result of this.

$$T_\theta T_R = \begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \quad T_\theta T_R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{T_\theta T_R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Example 4

Find the invariant point(s) under the transformation matrix

$$\begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}.$$

$$\begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} -x + 2y \\ 2x + 3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating components gives,

$$-x + 2y = x \Rightarrow x = y$$

$$2x + 3y = y \Rightarrow x = -y$$

Hence, $x = 0$ and $y = 0$.

$$\boxed{\text{Invariant point : } (0, 0)}$$

Example 5

Find the image of the curve $y = 3x^2$ under the transformation with matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Let a typical image point have coordinates (x', y'). Then,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Rightarrow \quad \underline{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3x' + 2y' \\ 2x' - y' \end{pmatrix}}$$

$$y = 3x^2$$

$$\therefore \quad \boxed{2x' - y' = 3(-3x' + 2y')^2}$$

AH Maths - MiA (2^{nd} Edn.)

- pg. 251-2 Ex. 13.10
  Q 1, 2 a, b, e, 3, 6 a, b, c.
- pg. 253-5 Ex. 13.11 Q 1, 2 a, b.
- pg. 255-6 Ex. 13.12 Q 1, 3, 4.

Ex. 13.10

1 In each of these, find the image of the point P under the transformation with the associated matrix A.

a P(1, 2); $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ b P(3, 4); $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ c P(−1, 2); $A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$

2 The triangle ABC has vertices A(1, 1), B(2, 3) and C(−1, 4).
Work out its image under the transformation associated with each of these matrices.

a $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reflection in $y = x$ b $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection in $x = 0$ e $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ enlargement ($\times$ 2)

3 A transformation maps (a, b) onto (a', b').

$(a, b) \rightarrow (a', b')$ where $a' = 2a + 3b$, $b' = a - 2b$

The matrix associated with such a mapping is $\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$.

In a similar way, find the matrix associated with each of these mappings.

a $(a, b) \rightarrow (a', b')$ where $a' = 2a + b$, $b' = -2a + b$

b $(a, b) \rightarrow (a', b')$ where $a' = 3a - 4b$, $b' = 2a + 3b$

c $(a, b) \rightarrow (5a + 3b, 3a - 2b)$

d $(a, b) \rightarrow (a + b, a - b)$

e $(a, b) \rightarrow (-b, -a)$

6 Describe in words the transformation associated with

a $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ b $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ c $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Ex. 13.11

1 Find all the invariant points of the transformation associated with these matrices.

a $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ b $B = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$ c $C = \begin{pmatrix} 2 & 1 \\ 7 & 8 \end{pmatrix}$ d $D = \begin{pmatrix} 4 & 5 \\ 3 & 6 \end{pmatrix}$ e $E = \begin{pmatrix} 3 & 4 \\ 3 & 7 \end{pmatrix}$

2 Find the equation of the image of each curve under the mapping with the given associated matrix.

a $y = x^2$, $A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$ b $y = 2x + 1$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

Ex. 13.12

1 Let X denote reflection in the x-axis, and Y denote reflection in the y-axis.

a Write the matrix associated with i X ii Y iii XY iv YX

b Write the image of P(a, b) after the transformation i XY ii YX

c Considering the geometry, is it a surprise that the image is the same in both cases?

3 Let F denote reflection in the x-axis, and G denote reflection in the line $y = x$.

a Find the matrix associated with i GF ii FG

b Explain briefly (geometrically) why $GF \neq FG$.

4 As already stated, the matrix $\begin{pmatrix} \cos\theta° & -\sin\theta° \\ \sin\theta° & \cos\theta° \end{pmatrix}$ is associated with an anticlockwise rotation of $\theta°$ about the origin.

a Write the matrices P, Q and R respectively associated with an anticlockwise rotation about the origin of i $A°$ ii $B°$ iii $(A + B)°$

b By expanding PQ and equating its entries with those of R, find well-known expansions for $\sin(A + B)$ and $\cos(A + B)$.
Explain why equating entries is valid.

Answers to AH Maths (MiA), pg. 251-2, Ex. 13.10

1 **a** $(1, 4)$ **b** $(7, 6)$ **c** $(-11, 2)$

2 **a** $A'(1, 1), B'(3, 2), C'(4, -1)$

b $A'(-1, 1), B'(-2, 3), C'(1, 4)$

e $A'(2, 2), B'(4, 6), C'(-2, 8)$

3 **a** $\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & -4 \\ 2 & 3 \end{pmatrix}$ **c** $\begin{pmatrix} 5 & 3 \\ 3 & -2 \end{pmatrix}$

d $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ **e** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

6 **a** Half-turn about the origin.

b Reflection in $y = -x$.

c Clockwise quarter-turn about the origin.

Answers to AH Maths (MiA), pg. 253-5, Ex. 13.11

1 **a** $x + y = 0$ **b** $x + 2y = 0$

c $x + y = 0$ **d** $3x + 5y = 0$

e $x + 2y = 0$

2 **a** $y = 2x^2 + 2x$ **b** $7x - 4y + 5 = 0$

Answers to AH Maths (MiA), pg. 255-5, Ex. 13.12

1 a i $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ii $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

iii $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ iv $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

b i $(-a, -b)$ ii $(-a, -b)$ **c** No

3 a i $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ii $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

b $FG = -GF$ (Quarter-turn in opposite direction)

4 a i $\begin{pmatrix} \cos A° & -\sin A° \\ \sin A° & \cos A° \end{pmatrix}$ ii $\begin{pmatrix} \cos B° & -\sin B° \\ \sin B° & \cos B° \end{pmatrix}$

iii $\begin{pmatrix} \cos (A+B)° & -\sin (A+B)° \\ \sin (A+B)° & \cos (A+B)° \end{pmatrix}$

b $\cos (A + B) = \cos A \cos B - \sin A \sin B$

$\sin (A + B) = \sin A \cos B + \cos A \sin B$

(The composition of two rotations is a rotation.)