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Unit 2 : Sequences and Series - Lesson 7

Maclaurin Series from First Principles

LI

- Know what a Maclaurin series is.
- Find Maclaurin series from scratch.

SC

- Differentiation.

The **Maclaurin series** (aka **Maclaurin expansion**) of a function f is a way of writing f as a power series of the form :

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} x^r = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots$$

where f is infinitely differentiable at 0

Some Standard Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

(valid for all $x \in \mathbb{R}$)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

(valid for all $x \in \mathbb{R}$)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

(valid for all $x \in \mathbb{R}$)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

(valid for $-1 < x \leq 1$)

Example 1

Find the first three non-zero terms of the Maclaurin expansion for $f(x) = \sin x$.

$$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$$

$$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$$

$$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -\cos 0 = -1$$

$$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = \sin 0 = 0$$

$$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = \cos 0 = 1$$

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} x^r$$

$$\therefore \sin x = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3$$

$$+ \frac{f^{(4)}(0)}{4!} x^4 + \frac{f^{(5)}(0)}{5!} x^5 + \dots$$

$$\Rightarrow \sin x = 0 + 1 \cdot x + \frac{0}{2} x^2 + \frac{(-1)}{6} x^3$$

$$+ \frac{0}{24} x^4 + \frac{1}{120} x^5 + \dots$$

$$\Rightarrow \boxed{\sin x = x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \dots}$$

Example 2

Find the Maclaurin series for $f(x) = (1 + x)^{1/2}$
up to the term in x^5 .

$$f(x) = (1 + x)^{1/2} \Rightarrow f(0) = 1$$

$$f'(x) = (1/2)(1 + x)^{-1/2} \Rightarrow f'(0) = 1/2$$

$$f''(x) = -(1/4)(1 + x)^{-3/2} \Rightarrow f''(0) = -1/4$$

$$f^{(3)}(x) = (3/8)(1 + x)^{-5/2} \Rightarrow f^{(3)}(0) = 3/8$$

$$f^{(4)}(x) = -(15/16)(1 + x)^{-7/2} \Rightarrow f^{(4)}(0) = -15/16$$

$$f^{(5)}(x) = (105/32)(1 + x)^{-9/2} \Rightarrow f^{(5)}(0) = 105/32$$

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} x^r$$

$$\begin{aligned} \therefore (1 + x)^{1/2} &= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 \\ &\quad + \frac{f^{(4)}(0)}{4!} x^4 + \frac{f^{(5)}(0)}{5!} x^5 + \dots \end{aligned}$$

$$\begin{aligned} \Rightarrow (1 + x)^{1/2} &= 1 + (1/2) \cdot x + \frac{(-1/4)}{2} x^2 + \frac{(3/8)}{6} x^3 \\ &\quad + \frac{(-15/16)}{24} x^4 + \frac{(105/32)}{120} x^5 + \dots \end{aligned}$$

$$\begin{aligned} \Rightarrow (1 + x)^{1/2} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \\ &\quad - \frac{5}{128}x^4 + \frac{7}{256}x^5 + \dots \end{aligned}$$

Questions

Find these Maclaurin expansions :

1) $f(x) = e^x$, up to x^5 .

2) $f(x) = (1 + x)^{-1}$, up to x^6 .

3) $f(x) = (1 - x)^{1/2}$, up to x^4 .

4) $f(x) = \cos x$, first four non-zero terms.

5) $f(x) = \ln(1 - x)$, up to x^5 .

6) $f(x) = \sec x$, first three non-zero terms.

Answers

$$1) e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$

$$2) (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \dots$$

$$3) (1 - x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \dots$$

$$4) \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

$$5) \ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$$

$$6) \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots$$