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Unit 1 : Differential Calculus - Lesson 7

Implicit Differentiation

LI

- Recognise Implicit Functions.
- Work out 1^{st} and 2^{nd} derivatives of Implicit Functions.

SC

- Product and Chain Rules.

Jan 10-12:11

A function f is given **explicitly** (f is an **explicit function**) if the output value y is given in terms of the input value.

An explicit function is recognised when y is given as a function of x.

Examples of Explicit Functions

$$y = \sin x$$

$$y = x^3$$

A function f is given **implicitly** (f is an **implicit function**) if the output value y is not given in terms of the input value.

An implicit function is usually identified when the variables x and y are mixed up in a higgledy-piggledy manner.

An implicitly defined function may or may not be solved for y.

Examples of Implicit Functions

$$y - \sin x = 0$$

$$x^2 + y^2 = 36$$

$$y \sin y + x = 4 \cos x$$

Useful Reminders/Things in Implicit Differentiation

- y is a function of x : $y(x)$.
- $\frac{d}{dx} y(x) = y'(x) = y'$.
- $\frac{d}{dx} y^2 = 2yy'$ (using the Chain Rule).
- $\frac{d}{dx} y^3 = 3y^2 y'$ (using the Chain Rule).
- $\frac{d}{dx}(xy)$ is differentiated using the Product Rule.
- The answer for y' will normally be a fraction involving both x and y.

Example 1

Find the gradient of the tangent at any point on the curve defined by $x^4 - 3xy + y^3 = 13$.

$$x^4 - 3xy + y^3 = 13$$

$$\therefore \quad 4x^3 - 3(1 . y + x . y') + 3y^2 y' = 0$$

$$\Rightarrow \quad 4x^3 - 3y - 3xy' + 3y^2 y' = 0$$

$$\Rightarrow \quad 4x^3 - 3y + y'(3y^2 - 3x) = 0$$

$$\Rightarrow \quad y'(3y^2 - 3x) = 3y - 4x^3$$

$$\Rightarrow \quad \boxed{y' = \frac{3y - 4x^3}{3y^2 - 3x}}$$

Example 2

Find the equation of the tangent to the curve defined by $xy + y^2 = 6$ at $y = 1$.

$$xy + y^2 = 6$$

$$\therefore \quad (1 . y + xy') + 2yy' = 0$$

$$\Rightarrow \quad y + xy' + 2yy' = 0$$

$$\Rightarrow \quad y + y'(x + 2y) = 0$$

$$\Rightarrow \quad y'(x + 2y) = -y$$

$$\Rightarrow \quad \underline{y' = -\frac{y}{x + 2y}}$$

To find the gradient, we need an (x, y) coordinate. $y = 1$ in the equation $xy + y^2 = 6$ gives $x = 5$.

$$\therefore \quad y'_{(5, 1)} = -\frac{1}{5 + 2}$$

$$\Rightarrow \quad \underline{y'_{(5, 1)} = -1/7 = m}$$

$$\therefore \quad y - 1 = -1/7\,(x - 5)$$

$$\Rightarrow \quad 7y - 7 = -x + 5$$

$$\Rightarrow \quad \boxed{x + 7y - 12 = 0}$$

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When finding the 2^{nd} derivative of an implicit function, it is better not to use the answer for the 1^{st} derivative and then to differentiate that (because this involves the quotient rule)

Instead, it is better to use a line of working in obtaining the 1^{st} derivative that has no fractional terms and differentiate that, rearrange and then solve for the 2^{nd} derivative

Example 3

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the function defined implicitly by $x^3 + 2xy = 7$.

Use the shorthand y' for $\frac{dy}{dx}$ and y'' for $\frac{d^2y}{dx^2}$.

$$x^3 + 2xy = 7$$

$$\therefore \quad 3x^2 + 2(1.y + xy') = 0$$

$$\Rightarrow \quad 3x^2 + 2y + 2xy' = 0 \quad \bigstar$$

$$\Rightarrow \quad 2xy' = -3x^2 - 2y$$

$$\Rightarrow \quad \boxed{y' = \frac{-(3x^2 + 2y)}{2x}}$$

Differentiate $\bigstar$ wrt x to get (after some simplification) :

$$6x + 4y' + 2xy'' = 0$$

$$\therefore \quad 6x - 4\left(\frac{3x^2 + 2y}{2x}\right) = -2xy''$$

$$\Rightarrow \quad \frac{12x^2 - 12x^2 - 8y}{2x} = -2xy''$$

$$\Rightarrow \quad \frac{-8y}{2x} = -2xy''$$

$$\Rightarrow \quad \boxed{y'' = \frac{2y}{x^2}}$$

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AH Maths - MiA (2^{nd} Edn.)

- pg. 89 Ex. 6.4 Q 1 - 11.
- pg. 90 Ex. 6.5 Q 1 b - g, 6, 7.

Ex. 6.4

1 Find $\frac{dy}{dx}$ for these implicit functions.

a $x^2 + 4xy + y^2 = 8$

b $x^2 = \ln y$

c $2x^2 + 2y^2 - 5x + 4y - 9 = 0$

d $x + 3 = e^y$

e $x^2 - xy + 3y^2 = 10$

f $x \tan y = e^x$

g $x^{\frac{2}{3}} + y^{\frac{2}{3}} = e^{\frac{2}{3}}$

h $\ln(x + y) = \tan^{-1} x$

i $5x^2 - 4xy + 3y^2 = 2$

j $\sin^{-1} x + \cos^{-1} y = 2x^3$

2 a i Given $x = \sin y$, find $\frac{dy}{dx}$ in terms of y.

ii Hence express $\frac{dy}{dx}$ in terms of x.

b Similarly deduce the derivative of

i $y = \cos^{-1} x$ ii $y = \tan^{-1} x$

3 Find an expression in terms of x and y for the gradient at the point (x, y) on the curve with equation $x^2 + y^2 = \frac{y}{x}$.

4 Prove that the curve $e^x + e^y - e = \frac{x}{y} + 1$ has a tangent at the point (0, 1) which is parallel to the x-axis.

5 Find the equation of the tangent to the curve with equation $xy^4 + 3x^2y^2 = 28$ at the point (1, 2).

6 Show that there is no point on the curve with equation $x + y = \ln(x - y)$ where the tangent is at 45° to the x-axis.

7 For the curve with equation $x \ln y = \cos x + \cos y$, show that the gradient at $x = 0$ is not defined.

8 Show that $x^2 = y^2 \ln y \Rightarrow \frac{dy}{dx} = \frac{2xy}{y^2 + 2x^2}$.

9 Find the equation of the tangent to the curve with equation $(x + 2y)^3 - 4x - 3y = 5$ at the point where it crosses the y-axis.

10 Find the gradient of the tangent at the point (e, e^2) on the curve given by $x \ln x + y \ln y = e(1 + 2e)$.

11 Show that $y = x - 1$ is the equation of the tangent at the point $\left(\frac{1}{2}, -\frac{1}{2}\right)$ to the curve with equation $\sin^{-1} x + \cos^{-1} y = \frac{5\pi}{6}$.

Ex. 6.5

1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y only, for each of these implicit functions.

a $x^2 + xy = 3$
b $x^3 + y^2 = 5$
c $\sqrt{x} + y^3 = 1$
d $xy = y^2 + 2$
e $(x + 1)(y - 1) = e^x$
f $xy^2 + 2 = y$
g $\ln(x + y) = x - y$
h $x^2 = y \ln y$
i $(x + y)^2 = e^y$
j $y = x(y + \sin x)$
k $y = \sin(x + y)$
l $x \sin^{-1} y = e^x$

6 For the function defined implicitly by $x^3 - xy + y^2 = 1$ evaluate $\frac{dy}{dx}$ at $(1, 1)$ and $\frac{d^2y}{dx^2}$ at $(1, 0)$.

7 For the function $y(x)$ defined implicitly by $y \cos x = e^x$ evaluate $y'\left(\frac{\pi}{3}\right)$ and $y''\left(\frac{\pi}{3}\right)$.

Answers to AH Maths (MiA), pg. 89, Ex. 6.4

1 a $-\dfrac{x+2y}{2x+y}$

b $2xy$

c $\dfrac{5-4x}{4(y+1)}$

d e^{-y}

e $\dfrac{y-2x}{6y-x}$

f $\dfrac{e^x-\tan y}{x\sec^2 y}$

g $-\left(\frac{y}{x}\right)^{\frac{3}{5}}$

h $\dfrac{x+y-x^2-1}{x^2+1}$

i $\dfrac{5x-2y}{2x-3y}$

j $\sqrt{\dfrac{1-y^2}{1-x^2}}-6x^2\sqrt{1-y^2}$

2 a i $\dfrac{1}{\cos y}$

ii $\dfrac{1}{\sqrt{1-x^2}}$

b i $-\dfrac{1}{\sin y}=-\dfrac{1}{\sqrt{1-x^2}}$

ii $\dfrac{1}{\sec^2 y}=\dfrac{1}{1+x^2}$

3 $\dfrac{2x^3+y}{x(1-2xy)}$

4 Differentiation gives $e^x+e^y\dfrac{dy}{dx}=\dfrac{y-\frac{dy}{dx}x}{y^2}$; $x=0, y=1$

$\Rightarrow \dfrac{dy}{dx}=0$

5 $10x+11y=32$

6 Differentiation gives $1+\dfrac{dy}{dx}=\dfrac{1-\frac{dy}{dx}}{x-y}$; $\dfrac{dy}{dx}=1\Rightarrow 2=0$

7 At $x=0$, $\cos y=-1$ and $\sin y=0$; differentiating:

$\dfrac{x}{y}\dfrac{dy}{dx}+\ln y=-\sin x-\sin y\dfrac{dy}{dx}$

At $x=0$, $\ln y=-\sin y\dfrac{dy}{dx}\Rightarrow\dfrac{dy}{dx}=\dfrac{\ln y}{-\sin y}=\dfrac{\ln y}{0}$ (undefined)

8 $x^2=y^2\ln y\Rightarrow \ln y=\dfrac{x^2}{y^2}\Rightarrow\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{y^2.2x-x^2.2y\frac{dy}{dx}}{y^4}$

$\Rightarrow\dfrac{dy}{dx}=\dfrac{2xy}{y^2+2x^2}.$

9 $8x+21y=21$

10 $-\frac{2}{3}$

11 The point lies on both curve and line.
The gradient at the point for line and curve is 1.

Answers to AH Maths (MiA), pg. 90, Ex. 6.5

1

b $-\frac{3x^2}{2y}, \frac{-3x(4y^2 + 3x^3)}{4y^3}$

c $-\frac{1}{6\sqrt{x}y^2}, \frac{3y^3 - 2\sqrt{x}}{36x\sqrt{x}y^5}$

d $\frac{y}{2y - x}, \frac{2y(y - x)}{(2y - x)^3}$

e $\frac{e^x - y + 1}{x + 1}, \frac{2(y - 1) + e^x(x - 1)}{(x + 1)^2}$

f $\frac{y^2}{1 - 2xy}, \frac{2y^3(2 - 3xy)}{(1 - 2xy)^3}$

g $\frac{x + y - 1}{x + y + 1}, \frac{4(x + y)}{(x + y + 1)^3}$

6 $-2, 18$

7 $2e^{\frac{\pi}{3}}(1 + \sqrt{3}), 4e^{\frac{\pi}{3}}(4 + \sqrt{3})$