Differential Calculus - Lesson 6

The Chain Rule

LI

- Differentiate a composition of functions.

SC

- The Chain Rule.

Reminder on Composition

If f and g are functions, the composition of f with g, usually written $f(g(x))$ ('f of g of x') is the function obtained by taking the output of g (which is $g(x)$) and using this as the input of f.

Generally speaking, $f(g(x))$ is not the same as $g(f(x))$.

Composition Example 1

If $f(x) = x^2$ and $g(x) = x + 3$, then :

$f(g(x)) = f(x + 3) = (x + 3)^2$ and

$g(f(x)) = g(x^2) = x^2 + 3$

Composition Example 2

If $f(x) = \sin x$ and $g(x) = x - 7$, then :

$f(g(x)) = f(x - 7) = \sin(x - 7)$ and

$g(f(x)) = g(\sin x) = (\sin x) - 7$

Composition Example 3

If $f(x) = x^3$ and $g(x) = \cos x$, then :

$f(g(x)) = f(\cos x) = (\cos x)^3 = \cos^3 x$ and

$g(f(x)) = g(x^3) = \cos(x^3)$

The Chain Rule

The Chain Rule tells us how to differentiate a composition of functions

Lagrange Form

If $y = f(g(x))$, then,

$$y' = f'(g(x)) \times g'(x)$$

(y dashed equals f dashed of g (x) multiplied by g dashed of x)

Leibniz Form

If $y = f(g(x))$, then letting $u = g(x)$, we have that $y = f(u)$ and $u = g(x)$ (i.e., y is a function of u and u is a function of x); then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example 1

Differentiate $y = (3x + 4)^5$.

$$y = (3x + 4)^5$$

$$\therefore \quad y' = 5(3x + 4)^{5-1} \cdot \frac{d}{dx}(3x + 4)$$

$$\Rightarrow \quad y' = 5(3x + 4)^4 \cdot 3$$

$$\Rightarrow \quad \boxed{y' = 15(3x + 4)^4}$$

Example 2

Differentiate $y = (7 - 2x)^6$.

$$y = (7 - 2x)^6$$

$$\therefore \quad y' = 6(7 - 2x)^{6-1} \cdot \frac{d}{dx}(7 - 2x)$$

$$\Rightarrow \quad y' = 6(7 - 2x)^5 \cdot (-2)$$

$$\Rightarrow \quad \boxed{y' = -12(7 - 2x)^5}$$

Example 3

Differentiate $y = \dfrac{1}{(8x + 1)^3}$.

$$y = \frac{1}{(8x + 1)^3}$$

$$y = (8x + 1)^{-3}$$

$$\therefore \quad y' = -3(8x + 1)^{-3-1} \cdot \frac{d}{dx}(8x + 1)$$

$$\Rightarrow \quad y' = -3(8x + 1)^{-4} \cdot 8$$

$$\Rightarrow \quad \boxed{y' = -24(8x + 1)^{-4} \quad \left(y' = -\frac{24}{(8x + 1)^4}\right)}$$

Example 4

Differentiate $y = \dfrac{1}{14\,(6x - 5)^4}$.

$$y = \frac{1}{14\,(6x - 5)^4}$$

$$y = \frac{1}{14}(6x - 5)^{-4}$$

$$\therefore \quad y' = -\frac{4}{14}(6x - 5)^{-4-1} \cdot \frac{d}{dx}(6x - 5)$$

$$\Rightarrow \quad y' = -\frac{4}{14}(6x - 5)^{-5} \cdot 6$$

$$\Rightarrow \quad y' = -\frac{12}{7}(6x - 5)^{-5}$$

$$\left(y' = -\frac{12}{7(6x - 5)^5} \right)$$

Example 5

Differentiate $y = (x^3 - 6x + 2)^9$.

$$y = (x^3 - 6x + 2)^9$$

$$\therefore \quad y' = 9(x^3 - 6x + 2)^8 \cdot \frac{d}{dx}(x^3 - 6x + 2)$$

$$\Rightarrow \quad y' = 9(x^3 - 6x + 2)^8 \cdot (3x^2 - 6)$$

$$\Rightarrow \quad \boxed{y' = 9(3x^2 - 6)(x^3 - 6x + 2)^8}$$

$$\boxed{y' = 27(x^2 - 2)(x^3 - 6x + 2)^8}$$

Example 6

Differentiate $y = \sqrt[4]{(17 - 2x)^3}$.

$$y = \sqrt[4]{(17 - 2x)^3}$$

$$y = (17 - 2x)^{3/4}$$

$$\therefore \quad y' = \frac{3}{4}(17 - 2x)^{3/4 - 1} \cdot \frac{d}{dx}(17 - 2x)$$

$$\Rightarrow \quad y' = \frac{3}{4}(17 - 2x)^{-1/4} \cdot (-2)$$

$$\Rightarrow \quad y' = -\frac{3}{2}(17 - 2x)^{-1/4}$$

$$\left(y' = -\frac{3}{2(17 - 2x)^{1/4}} \right)$$

$$\left(y' = -\frac{3}{2\sqrt[4]{17 - 2x}} \right)$$

Example 7

Differentiate $y = 4 \cos(2x)$.

$$y = 4 \cos(2x)$$

$$\therefore \quad y' = 4(-\sin(2x)) \cdot \frac{d}{dx}(2x)$$

$$\Rightarrow \quad y' = -4 \sin(2x) \cdot 2$$

$$\Rightarrow \quad \boxed{y' = -8 \sin(2x)}$$

Example 8

Differentiate $y = \sin(3x + 2)$.

$$y = \sin(3x + 2)$$

$$\therefore \quad y' = \cos(3x + 2) \cdot \frac{d}{dx}(3x + 2)$$

$$\Rightarrow \quad y' = \cos(3x + 2) \cdot 3$$

$$\Rightarrow \quad \boxed{y' = 3\cos(3x + 2)}$$

Example 9

Differentiate $y = \sin^5 x$.

$$y = \sin^5 x$$

$$y = (\sin x)^5$$

$$\therefore \quad y' = 5(\sin x)^{5-1} \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \quad y' = 5(\sin x)^4 \cdot \cos x$$

$$\Rightarrow \quad \boxed{y' = 5\sin^4 x \cos x}$$

Example 10

Differentiate $y = \frac{1}{\cos x}$.

$$y = \frac{1}{\cos x}$$

$$y = (\cos x)^{-1}$$

$$\therefore \quad y' = -(\cos x)^{-1-1} \cdot \frac{d}{dx}(\cos x)$$

$$\Rightarrow \quad y' = -(\cos x)^{-2} \cdot (-\sin x)$$

$$\Rightarrow \quad \boxed{y' = (\cos x)^{-2} \sin x \quad \left(y' = \frac{\sin x}{\cos^2 x} \right)}$$

Example 11

Differentiate $y = \sin(\cos x)$.

$$y = \sin(\cos x)$$

$$\therefore \quad y' = \cos(\cos x) \cdot \frac{d}{dx}(\cos x)$$

$$\Rightarrow \quad y' = \cos(\cos x) \cdot (-\sin x)$$

$$\Rightarrow \quad \boxed{y' = -\sin x \cos(\cos x)}$$

CfE Higher Maths

- pg. 232 - 234 Ex. 9G
  Q 1 - 4, 5 i-q, 6

- pg. 235 - 237 Ex. 9H
  Q 1 - 5, 6 a-d, 13 d

Questions

1 Differentiate with respect to x

a $(x+4)^3$ **b** $(x-2)^6$ **c** $(x+3)^9$ **d** $(x-1)^5$

e $5(x+1)^4$ **f** $8(x-3)^6$ **g** $\frac{1}{2}(x+5)^8$ **h** $\frac{2}{7}(x-5)^7$

i $(x+2)^{-1}$ **j** $(x-5)^{-4}$ **k** $(x+6)^{-7}$ **l** $(x-3)^{-4}$

m $4(x-2)^{-3}$ **n** $9(x-7)^{-2}$ **o** $\frac{3}{4}(x+1)^{-8}$ **p** $\frac{5}{6}(x-4)^{-9}$

q $6+(x-1)^4$ **r** $3-(x+4)^5$ **s** $2(x+2)^9-4x^3$

t $6\sqrt{x}+5(x-1)^4$ **u** $3(x+4)^{-1}-\frac{5}{x^2}$ **v** $\frac{4}{7x^3}+8(x-4)^{-2}$

2 Differentiate with respect to x

a $(3x+1)^6$ **b** $(5x-2)^4$ **c** $(2x-7)^5$ **d** $(4x+1)^9$

e $2(3x-4)^7$ **f** $10(6x+2)^3$ **g** $8(5x-4)^8$ **h** $6(7x-1)^4$

i $(4x-1)^{-2}$ **j** $(2x+5)^{-8}$ **k** $(9x-2)^{-1}$ **l** $(5x+4)^{-6}$

m $3(2x-1)^{-4}$ **n** $5(7x+1)^{-2}$ **o** $(2x+5)^{-3}-8x$

p $\frac{1}{\sqrt[4]{x^5}}-4(3x-1)^{-1}$ **q** $(x-4)^{-6}+\frac{4x^2-1}{x}$ **r** $3(8x-1)^{-2}-\frac{4}{\sqrt{x}}$

s $\sqrt{x}\left(2x-\frac{1}{x^2}\right)+6(4x-1)^{-2}$ **t** $-2(3x+4)^{-5}-\frac{2}{5x^{10}}$

3 Differentiate with respect to x

a $(1-x)^5$ **b** $(5+x)^{-3}$ **c** $(3+7x)^4$ **d** $\left(\frac{2}{3}x-4\right)^6$

e $(2-5x)^6$ **f** $\left(\frac{3x}{5}+2\right)^{10}$ **g** $\frac{(2-3x)^4}{5}$ **h** $(6-x)^{-1}$

i $5(1-2x)^{-3}-7x^2$ **j** $6x\sqrt{x}-\frac{(1-2x)^{-4}}{8}$

4 Differentiate with respect to x

a $\frac{1}{(x+1)^4}$ **b** $\frac{1}{(x-5)^2}$ **c** $\frac{1}{(4x+1)^5}$ **d** $\frac{1}{3x-4}$

e $\frac{5}{x-3}$ **f** $\frac{2}{(x+1)^3}$ **g** $\frac{6}{(2x-5)^2}$ **h** $\frac{8}{(5x-1)^6}$

i $\frac{1}{3(x+2)}$ **j** $\frac{1}{4(x-1)^2}$ **k** $\frac{3}{2(x+1)^4}$ **l** $\frac{2}{9(3x-2)^5}$

m $\frac{1}{2-x}$ **n** $\frac{8}{5-3x}$ **o** $\frac{1}{(5-8x)^2}$ **p** $\frac{9}{4(1-2x)^2}$

5 Differentiate with respect to x

i $\frac{1}{(x-1)^{\frac{2}{3}}}$ **j** $\frac{1}{4(x+2)^{\frac{3}{2}}}$ **k** $\sqrt[3]{(x-4)^2}$ **l** $\sqrt{(6x+1)^5}$

m $\frac{1}{\sqrt{x+2}}$ **n** $\frac{8}{\sqrt[4]{(x-3)^3}}$ **o** $\frac{1}{\sqrt[5]{4-x}}$ **p** $6\left(\sqrt{(2-5x)^3}\right.$

q $\frac{4}{\sqrt[4]{(5-2x)^3}}$

6 Differentiate with respect to x

a $(x^2-3)^4$ **b** $(x^3-2x^2+1)^5$ **c** $(x^4-5x-2)^3$

d $(4-3x^2)^6$ **e** $(2x^2+5x-3)^{-1}$ **f** $(3-2x-x^3)^{-4}$

g $\frac{1}{x^2-5}$ **h** $\sqrt{2x^2-x+5}$ **i** $\frac{1}{\sqrt{2-3x^3}}$

j $\sqrt{\left(x^4-1\right)^3}$ **k** $\sqrt[3]{\left(x^3+x^2+x+1\right)}$ **l** $\left(\sqrt{x}-2\right)^6$

Answers

1 **a** $3(x+4)^2$

b $6(x-2)^5$

c $9(x+3)^8$

d $5(x-1)^4$

e $20(x+1)^3$

f $48(x-3)^5$

g $4(x+5)^7$

h $2(x-5)^6$

i $-\frac{1}{(x+2)^2}$

j $-\frac{4}{(x-5)^5}$

k $-\frac{7}{(x+6)^8}$

l $-\frac{4}{(x-3)^5}$

m $-\frac{12}{(x-2)^4}$

n $-\frac{18}{(x-7)^3}$

o $-\frac{6}{(x+1)^9}$

p $-\frac{15}{2(x-4)^{10}}$

q $4(x-1)^3$

r $-5(x+4)^4$

s $18(x+2)^8 - 12x^2$

t $20(x-1)^3 + \frac{3}{\sqrt{x}}$

u $\frac{10}{x^3} - \frac{3}{(x+4)^2}$

v $-\frac{12}{7x^4} - \frac{16}{(x-4)^3}$

2 **a** $18(3x+1)^5$

b $20(5x-2)^3$

c $10(2x-7)^4$

d $36(4x+1)^8$

e $42(3x-4)^6$

f $180(6x+2)^2$

g $320(5x-4)^7$

h $168(7x-1)^3$

i $-\frac{8}{(4x-1)^3}$

j $-\frac{16}{(2x+5)^9}$

k $-\frac{9}{(9x-2)^2}$

l $-\frac{30}{(5x+4)^7}$

m $-\frac{24}{(2x-1)^5}$

n $-\frac{70}{(7x+1)^3}$

o $-\frac{6}{(2x+5)^4} - 8$

p $-\frac{5}{4x^{\frac{9}{4}}} + \frac{12}{(3x-1)^2}$

q $4 - \frac{6}{(x-4)^7} + \frac{1}{x^2}$

r $\frac{2}{x^{\frac{3}{2}}} - \frac{48}{(8x-1)^3}$

s $\frac{3}{2x^{\frac{5}{2}}} + 3\sqrt{x} - \frac{48}{(4x-1)^3}$

t $\frac{4}{x^{11}} + \frac{30}{(3x+4)^6}$

3 **a** $-5(1-x)^4$

b $-\frac{3}{(x+5)^4}$

c $28(7x+3)^3$

d $4\left(\frac{2x}{3} - 4\right)^5$

e $-30(2-5x)^5$

f $6\left(2 + \frac{3x}{5}\right)^9$

g $-\frac{12}{5}(2-3x)^3$

h $\frac{1}{(6-x)^2}$

i $\frac{30}{(1-2x)^4} - 14x$

j $9\sqrt{x} - \frac{1}{(1-2x)^5}$

4 **a** $-\frac{4}{(x+1)^5}$

b $-\frac{2}{(x-5)^3}$

c $-\frac{20}{(4x+1)^6}$

d $-\frac{3}{(3x-4)^2}$

e $-\frac{5}{(x-3)^2}$

f $-\frac{6}{(x+1)^4}$

g $-\frac{24}{(2x-5)^3}$

h $-\frac{240}{(5x-1)^7}$

i $-\frac{1}{3(x+2)^2}$

j $-\frac{1}{2(x-1)^3}$

k $-\frac{6}{(x+1)^5}$

l $-\frac{10}{3(3x-2)^6}$

m $\frac{1}{(2-x)^2}$

n $\frac{24}{(5-3x)^2}$

o $\frac{16}{(5-8x)^3}$

p $\frac{9}{(1-2x)^3}$

5 **i** $-\frac{2}{3(x-1)^{\frac{5}{3}}}$

j $-\frac{3}{8(x+2)^{\frac{5}{2}}}$

k $\frac{2}{3(x-4)^{\frac{1}{3}}}$

l $15(6x+1)^{\frac{3}{2}}$

m $-\frac{1}{2(x+2)^{\frac{3}{2}}}$

n $-\frac{6}{(x-3)^{\frac{7}{4}}}$

o $\frac{1}{5(4-x)^{\frac{6}{5}}}$

p $-45(2-5x)^{\frac{1}{2}}$

q $\frac{6}{(5-2x)^{\frac{7}{4}}}$

6 **a** $8x\left(x^2-3\right)^3$

b $5\left(3x^2-4x\right)\left(x^3-2x^2+1\right)^4$

c $3\left(4x^3-5\right)\left(x^4-5x-2\right)^2$

d $-36x\left(4-3x^2\right)^5$

e $-\frac{4x+5}{\left(2x^2+5x-3\right)^2}$

f $-\frac{4\left(-2-3x^2\right)}{\left(-x^3-2x+3\right)^5}$

g $-\frac{2x}{\left(x^2-5\right)^2}$

h $\frac{4x-1}{2\sqrt{2x^2-x+5}}$

i $\frac{9x^2}{2\left(2-3x^3\right)^{\frac{3}{2}}}$

j $6x^3\sqrt{x^4-1}$

k $\frac{3x^2+2x+1}{3\left(x^3+x^2+x+1\right)^{\frac{2}{3}}}$

l $\frac{3\left(\sqrt{x}-2\right)^5}{\sqrt{x}}$

Questions

1 Differentiate with respect to x

a $\sin 2x$ **b** $\cos 5x$ **c** $3\sin 4x$

d $6\cos 3x$ **e** $\cos\left(2x + \frac{\pi}{6}\right)$ **f** $\sin(6x - \pi)$

g $\frac{3}{2}\sin 4x$ **h** $\frac{3}{5}\cos 10x$ **i** $\frac{1}{3}\sin(6x + 2)$

j $-8\cos\left(\frac{1}{4}x\right)$ **k** $\sin(2 - x)$ **l** $-\frac{1}{3}\sin(1 - 9x)$

m $\cos 5x - 2\sin 3x$ **n** $\sin x + 8\cos 3x$ **o** $\sin 3x - 2\cos x$

p $\frac{5}{4}\sin(2x - \pi) - \cos\left(\frac{3}{2}x\right)$

2 **a** Find the gradient of the tangent to the curve $y = \sin 2x$ at the point where $x = \frac{\pi}{6}$

b The function f is defined by $f(x) = 2\cos\left(x - \frac{\pi}{2}\right)$ Find $f'\left(\frac{2\pi}{3}\right)$

c A curve has equation $y = \cos 3x$. Find the gradient of the curve at the point where $x = \frac{2\pi}{9}$

d The function g is defined by $g(x) = 4\sin\left(\frac{1}{2}x\right)$. Find the rate of change of g when $x = \frac{\pi}{2}$

e Given $y = 6\cos\left(2x + \frac{\pi}{6}\right)$, find $\frac{dy}{dx}$ when $x = \frac{3\pi}{2}$

3 **You will need a calculator for this question**. Give your answers to 2 decimal places.

a Given $y = \sin 5x$, find $\frac{dy}{dx}$ when $x = 1$.

b The function h is defined by $h(x) = 5\cos(3x - 1)$. Evaluate $h'(-3)$.

c Find the gradient of the tangent to the curve with equation $y = 2\sin(5 - 3x)$ at the point where $x = \frac{\pi}{2}$

d The function f is defined by $f(x) = 2\cos(\pi - 3x)$. Find the rate of change of f when $x = -2$.

4 Differentiate with respect to x.

a $\sin^2 x$ **b** $\cos^3 x$ **c** $2\sin^3 x$

d $5\cos^6 x$ **e** $-\frac{1}{2}\sin^4 x$ **f** $\frac{6}{5}\cos^5 x$

g $(\sin x + \cos x)^3$ **h** $(\sin(2x - 1))^5$

5 Find the gradient of the tangent to the curve with equation $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$

6 Differentiate with respect to x.

a $\frac{1}{\sin x}$ **b** $\frac{3}{\cos x}$ **c** $\sqrt{\sin x}$ **d** $\frac{1}{\cos^2 x}$

13 Functions f and g are defined by $f(x) = \cos x$ and $g(x) = 3x^2 - 1$

d At the point where $x = \frac{2\pi}{3}$, the tangent to the curve $y = g(f(x))$ is parallel to the straight line $ax - \sqrt{3}y = 0$. Determine algebraically the value of a.

Answers

1 **a** $2\cos 2x$
b $-5\sin 5x$
c $12\cos 4x$
d $-18\sin 3x$
e $-2\sin\left(2x + \frac{\pi}{6}\right)$
f $\cos(6x - \pi)$
g $6\cos(4x)$
h $-6\sin(10x)$
i $2\cos(6x + 2)$
j $2\sin\left(\frac{x}{4}\right)$
k $-\cos(2 - x)$
l $3\cos(1 - 9x)$
m $-6\cos(3x) - 5\sin(5x)$
n $\cos x - 24\sin(3x)$
o $3\cos 3x + 2\sin x$
p $\frac{5}{2}\cos(2x - \pi) + \frac{3}{2}\sin\left(\frac{3x}{2}\right)$

2 **a** 1
b -1
c $-\frac{3\sqrt{3}}{2}$
d $\sqrt{2}$
e 6

3 **a** 1.42
b -8.16
c -5.75
d 1.68

4 **a** $2\cos x \sin x$
b $-3\sin x(\cos x)^2$
c $6\cos x(\sin x)^2$
d $-30\sin x(\cos x)^5$
e $-2\cos x(\sin x)^3$
f $-6\sin x(\cos x)^4$
g $3(\cos x - \sin x)(\cos x + \sin x)^2$
h $10\cos(1 - 2x)(\sin(2x - 1))^4$

5 1

6 **a** $-\frac{\cos x}{\sin x}\frac{1}{\sin x} = -\frac{\cos x}{(\sin x)^2}$
b $3\frac{\tan x}{\cos x}$
c $\frac{\cos x}{2\sqrt{\sin x}}$
d $2\frac{\tan x}{(\cos x)^2}$

13 **d** $a = 4.5$